DEVELOPING PROPORTIONAL REASONING IN MIDDLE SCHOOL STUDENTS

by

Marcie Beck McIntosh

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SUPERVISORY COMMITTEE APPROVAL

of a project submitted by

MARCIE B MCINTOSH

This project has been read by each member of the following supervisory committee and by majority vote has been found to be satisfactory.

Date ________________

Committee Chair: Emina Alibegovic

Date ________________

Committee Member: Aaron Bertram

Date ________________

Committee Member: Hugo Rossi
# TABLE OF CONTENTS

ABSTRACT .......................................................................................................................... 4
ACKNOWLEDGEMENTS ..................................................................................................... 5
1. INTRODUCTION ........................................................................................................... 6
2. WHAT IS PROPORTIONAL REASONING? ................................................................. 7
3. COMPONENTS IN DEVELOPING PROPORTIONAL REASONING ...................... 9
   3.1 NCTM ESSENTIAL TEACHER UNDERSTANDINGS ........................................... 9
   3.2 RELATIVE THINKING ......................................................................................... 11
   3.3 UNITS AND UNITIZING ..................................................................................... 15
   3.4 PARTITIONING ................................................................................................... 18
   3.5 RATIONAL NUMBER INTERPRETATIONS ......................................................... 20
       AS PART/WHOLE COMPARISONS ..................................................................... 21
       AS OPERATORS .................................................................................................. 22
       AS QUOTIENTS ................................................................................................. 23
       AS MEASURES .................................................................................................. 24
       AS RATIOS ......................................................................................................... 24
   3.6 RATIO SENSE ....................................................................................................... 26
   3.7 ATTENDING TO QUANTITIES AND CHANGE .................................................... 31
4. MCF RESEARCH FINDINGS .......................................................................................... 35
5. DEVELOPING PROPORTIONAL REASONING THROUGH TASKS .................. 43
   5.1 CONCEPTUALIZE BEFORE COMPUTING ......................................................... 45
   5.2 EXPLICITLY HIGHLIGHT MULTIPLICATIVE UNDERSTANDING .................. 46
   5.3 UTILIZE DIVERSE CONTEXTS, SOLUTION METHODS AND UNITS .......... 48
   5.4 PROVIDE STUDENTS OPPORTUNITIES TO TEST THEIR THINKING ........ 49
6. PERSONAL REFLECTIONS ......................................................................................... 50
REFERENCES ...................................................................................................................... 53
ABSTRACT

Proportional reasoning may be the most commonly used mathematics in the world and yet it is estimated that more than half the adult population cannot reason proportionally. This project reviews the current literature in the Multiplicative Conceptual Field (MCF) to characterize and define proportional reasoning. The research indicates that proportional reasoning is developed within a web of content understandings. This project articulates the cognitive processes involved in developing relative thinking, partitioning, unitizing, rational number interpretations, ratio sense and attention to quantities and change. In addition several specific MCF research projects are highlighted. Pedagogical implications are articulated throughout the project and a summary of the author’s personal “big ideas” from the research are discussed.
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1. INTRODUCTION

Proportions may be the most commonly applied mathematics in the real world (Hoffer and Hoffer, 1988) and yet it is estimated that more than half the adult population cannot reason proportionally (Lamon, 1999). Middle school students are instructed in sixth-, seventh- and eighth-grade mathematics usually with pre-algebra as the identifying theme. This nomenclature seems to identify the only purpose of middle school mathematics as a precursor to “the real math” of algebra. However consensus is building that the mathematical development taking place in the middle school grades constitutes the most important set of developments in a person’s mathematical schooling (Usiskin, 2004). During these years, students must make several major transitions in their mathematical thinking. A central change in thinking is required in a shift from natural number to rational numbers and from additive concepts to multiplicative concepts.

“Mastery of many of the number concepts and number relationships in the middle grades appears to require a reconceptualization of number, a significant change from the primary grades in the way number is conceived…..Multiplication is not simply repeated addition, and rational numbers are not simply ordered pairs of whole numbers. The new concepts are not the sums of previous ones. Competency with middle school number concepts requires a break with simpler concepts of the past and a reconceptualization of number itself (Hiebert and Behr, 1988)

Researchers have stated that proportional reasoning involves watershed concepts: that are at the cornerstone of higher mathematics (Lamon 1999), are the applied mathematics of real life and much more complex and interconnected than previously thought (Vergnaud, 1994). This complexity has led educational researchers to combine the study of concepts such as fraction, ratio, rate, percentage and proportion into a multiplicative conceptual field (MCF) of study.

“The topics included under the rubric “multiplicative conceptual field” (MCF) possess an interconnectedness and complexity that presents a unique research challenge and teaching challenge. Research projects in the MCF field attempt to consider seriously the difficulties and challenges students and teachers face when approaching the topics of multiplication, division,
ratio and proportion and to find new ways to think and talk about these difficulties and challenges” (Harel, 1994).

Thus multiplicative reasoning is the foundation of proportional reasoning and proportionality may be the unifying theme needed to highlight the important mathematics of the middle school years (Lanius, 2003).

The purpose of this project is to increase teacher understanding of developing proportional reasoning in two ways: (1) By examining the current literature to determine what is meant by proportional reasoning and what related concepts are essential in developing student proficiency. (2) By reviewing MCF research for important pedagogical findings that highlight best practices in teaching methods.

2. WHAT IS PROPORTIONAL REASONING?

According to Van de Walle (2010) “proportional reasoning is difficult to define in a simple sentence or two. It is not something that you either can or cannot do. It is both a qualitative and quantitative process.” In the progressions of the common core state standards for ratios and proportional relationship objectives for Grades 6-7 an appendix devoted to giving definitions of the terms ratio, rate and proportional relationships is included because many different authors have used these terminologies in widely differing ways. It may be easier to say what proportional reasoning is not than to give a precise definition. Proportional reasoning is NOT the ability to set up a proportion and apply cross multiplication to find a missing number. Proportional reasoning is not usually involved when students apply memorized rules or algorithms (Lamon, 1999).

Proportional reasoning involves the deliberate use of multiplicative relationships to compare quantities and to predict the value of one quantity based on the values of another. The term *deliberate* is used to clarify that proportional reasoning is more about the use of number
sense than formal, procedural solving of proportions (Service Ontarios, 2013). The essence of proportional reasoning is the consideration of quantities in relative terms rather than absolute terms, and a shift from additive reasoning to multiplicative reasoning. Reasoning involves numerous understandings, including grasping the meaning of a ratio as a multiplicative comparison and as a composed unit, making connections among ratios, fractions, and quotients and understanding the ideas involved in increasingly complex situations. Developing the” big idea” of proportionality and its associated understanding is not easy (Lobato, 2012).

The operating theory for instruction is that by providing students experiences with some of the critical components of proportional reasoning before proceeding to more abstract formal presentations, teachers increase students’ chances of developing proportional reasoning. To achieve proportional reasoning one needs extensive time and experience to build up many kinds of knowledge: concepts, ways of thinking and ways of acting (operations)—all linked to appropriate contexts (Lamon, 1999). The multiplicative conceptual field affirms there must be a tightly connected web of ideas on multiplicative rather than additive ideas. The more connections made between topics and contexts, the more proportional reasoning improves.

Research has shown six known content areas that contribute to developing proportional reasoning: See Figure 3.1

Based on the research, NCTM published Developing Essential Understanding of Ratios, Proportions & Proportional Reasoning (Lobito, 2010) that identifies 10 essential understandings for teachers. An examination of these recommendations, the six content areas (relative thinking, partition, unitizing, rational number interpretations, ratio sense and attention to quantities and change) and MFC research will deepen our comprehension of what proportional reasoning entails.
3. COMPONENTS IN DEVELOPING PROPORTIONAL REASONING

3.1 NCTM ESSENTIAL TEACHER UNDERSTANDINGS

Lobato (2010) states that the BIG IDEA of proportionality is “when two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor”. Although this big idea can be easily stated, developing understanding is a complex process for both students and teachers. In an effort to provide clarity, ten essential understandings for teachers has been clearly articulated:

“Essential Understandings

1. Reasoning with ratios involves attending to and coordinating two quantities.
2. A ratio is a multiplicative comparison of two quantities, or it is a joining of two quantities in a composed unit.
3. Forming a ratio as a measure of a real world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.
4. A number of mathematical connections link ratios and fractions:
   a. Ratios are often expressed in fraction notation, although ratios and fractions do not have identical meaning.
   b. Ratios are often used to make “part-part” comparisons but fractions are not.
   c. Ratios and fractions can be thought of as overlapping sets.
d. Ratios can often be meaningfully reinterpreted as fractions.
5. Ratios can be meaningfully reinterpreted as quotients
6. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.
7. Proportional reasoning is complex and involves understanding that –
   a. Equivalent ratios can be created by iterating and/or partitioning a composed unit;
   b. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and
   c. The two types of ratios—composed units and multiplicative comparisons—are related.
8. A rate is a set of infinitely many equivalent ratios.
9. Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.
10. Superficial cues present in the context of a problem do not provide sufficient evidence of proportional relationships between quantities.” (Labato, 2010)

These teacher understandings must not only be internalized but used to guide students through a progression of four cognitive shifts. First, students need to make a transition from focusing on only one quantity to realizing that two quantities are important. Second, students need to move from making additive comparisons to forming ratios between two quantities. Third, students must shift from using composed-unit strategies to also making and using multiplicative comparisons. And the final transition is from developing a few “easy” equivalent ratios to creating a set of infinitely many equivalent ratios. These shifts do not outline a teaching progression but highlight how teachers can assess students’ levels of proportional reasoning.

Langrall (2000) also articulates four essential components of student understanding for proportional reasoning. Students need to (1) recognize the difference between absolute and relative change, (2) recognize when using a ratio is reasonable, (3) understand covariance to mean that the relationship remains unchanged (invariant) when the quantities change and (4) develop the ability to “unitize.” Articulating the necessary progression of cognitive development in students, helps teachers identify the foundational understanding instruction
should be built on. These foundational understanding can best be examined in light of the six identified content areas.

### 3.2 RELATIVE THINKING

One of the most important types of thinking required for proportional reasoning is the ability to analyze change in relative terms. This ability requires students to understand and identify the differences between absolute change (the actual amount of change independent of and unrelated to anything else) and relative change (how much something changed in comparison to something else). Making comparisons and thinking in terms of two quantities is the essential understanding underpinning proportional reasoning and is the foundation of recognizing relative change. Relative thinking is a cognitive skill that can best taught in contexts. A growth example illustrates the difference between absolute and relative change. Two snakes, one starting at a length of 4 feet and the other at a length of six feet grow to be 8 and 10 feet long respectively. How much did they change? In absolute terms both snakes changed the same amount. They each are longer by 4 feet. In relative terms however, the first snake has grown twice as long while the second has only grown 1 and 2/3 as long. The relative change is much larger for the smaller snake. The question “How much did they change?” is not specific in the type of change being requested. However, almost universally people respond with the additive change unless directly prompted to make a comparison. Helping students understand that change can be expressed in two ways draws attention to the distinction between additive and multiplicative thinking.

Relative thinking attaches significance or meaning to a particular quantity by comparing it to another quantity. It is the basis for ratios and comparative thinking. A person cannot grasp the notion of a ratio if they can only recognize absolute changes. Relative thinking should begin in initial fraction instruction. Fractions as a part-whole are a special type of ratio. In fraction
instruction, relative thinking can be highlighted in 3 ways by emphasizing (1) the relationship between the size of pieces and the number of pieces, (2) the need to compare fractions relative to the same unit, and (3) the concept of equivalent fractional representations. The idea that a single fractional representation can be found in any color, size, shape, orientation, location, partition etc. highlights the relative amount rather than absolute measurement of fractions.

Percent problems are the prime example of relative thinking but are often not explicitly taught as such. Often they are incomprehensible to students because students do not recognize the difference between the questions “how many?” and “how much of?”. This distinction can be explicitly taught by giving students opportunities to answer both questions in the context of one problem scenario. Example 1: Mr. Thompson took 3 vacation days this week. How many days did he work? How much of the week did he work? Example 2: The number of students in piano increased from 12 to 15: How many additional students joined piano? How much of an increase was the change? Asking the questions this way makes the difference between an absolute and relative change apparent. This distinction between how many and how much of is not always as clear as in the above examples. Students need to be taught that many questions can be interpreted both ways like the question in the snake example: How much did it change? By working through examples such as these, students begin to understand that relative change does not give an amount of change but a “times” bigger or smaller (a scale or percent of change like three sevenths of a week or 25% more students).

Ratios imply thinking multiplicatively and are explicitly a comparative index. This multiplicity is often missed by students as many ratio tables allow students to use additive (or absolute) comparisons with building up/down strategies to solve ratio problems. Ratio problems need to be structured so that the answers are not always nice numbers that yield themselves to using both strategies. This development of relative thinking is a cognitive process.
which can be explicitly taught by comparing and contrasting additive versus multiplicative methods in numerous contexts.

**Teaching Implications:**

Example 3.2.1 and 3.2.2 Which is most?

To develop relative thinking teachers can: Make explicit the difference between absolute and relative comparisons by presenting students with the absolute-relative choices (see Example 3.2.1 and 3.2.2). Choose ratio problems carefully to highlight multiplicative rather than additive thinking. Compare and contrast different methods of solving ratios. Explicitly show the difference in solving ratio problems both ways (see Example 3.2.3). Choose ratio problems carefully to highlight multiplicative rather than additive thinking. Compare and contrast different methods of solving ratios. Be conscious of the way in which you ask questions (see Example 3.2.4).
Example 3.2.3  Additive versus multiplicative ratio solution

To make fruit punch, use two cups of juice for every 3 cups of apple juice. If you already have 12 cups of apple juice, how many cups of grape juice will you need to make fruit punch?

**ADDITIVE SOLUTION**

<table>
<thead>
<tr>
<th>Cups of Grape Juice</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cups of Apple Juice</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

**MULTIPLICATIVE SOLUTION**

<table>
<thead>
<tr>
<th>Cups of Grape Juice</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cups of Apple Juice</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>

If you change the question to 200 cups or 1 cup of apple juice, multiplication becomes the preferred option rather than an alternative solution method.
Example 3.2.4 How you ask the question.

1. Your favorite store is having a sale. Is it more helpful to you to know that an item is $2 off or 20% off?
2. What kind of information is necessary to describe the “crowdedness” of an elevator?
3. Which of the following rectangles is most square: 42’ by 84’, 200’ by 242’ or 356’ by 398’. Why?

To assess student learning ask open ended questions and see how students respond. See if they look for both absolute and relative comparisons. Ask for two answers or two ways to find a solution. Have students identify what type of comparisons they are making absolute (additive) or relative (multiplicative).

“If students recognize and understand the difference between the additive and multiplicative approaches, this is a beginning to being able to reason proportionally.” (Van de Walle, 2010)

3.3 UNITS AND UNITIZING

The questions of “How many?” or “How much of?” require units of measurement. If you measure the same “stuff” with different size cups the measurement will be different, larger or smaller, depending on the unit. Implicit in our understanding of change is the measurement we are using. Think of time. Are we measuring in hours, minutes or seconds? What is a unit? The meaning in a fraction is derived from the context either implicitly or explicitly and sometimes the problem could involve several units. Example 3.3.1 could represent 3/5, 2/3, 5/3, 3/2, 2/5 or 5/2 depending on the “unit.” If the unit is the whole rectangle then the 3 dark sections represent 3/5. If the unit is composed of the 3 dark rectangles then the white sections represent 2/3 and the 5 rectangles combined represent 5/3. If the two white sections are the unit, then the dark rectangles represent 3/2 and the 5 combined rectangles represent 5/2.

Looking for and identifying the unit is fundamental to understanding the context of a problem.
but is often overlooked. Too often students are asked questions about situations like Example 3.3.1 without any thought to different interpretations. Units that are implicit should be made explicit either by the teacher or by the student.

Unitizing is a cognitive process that occurs after identifying the unit but allows subjective preference for the mental size you prefer to think about. It is the ability to conceptualize a unit in terms of many different sized pieces. In mathematics the ability to conceive of a commodity in terms of more than one size chunk frequently adds convenience, simplicity, speed and sophistication to one’s mathematical reasoning. Textbooks rarely use this approach when they teach ratios and rates. For example consider the following problem: 3 oranges for $1.20 and you want 9. A standard unit approach would calculate that each orange is $0.40 and then multiply by 9 oranges to find the final price. The convenient unitizing is in bundles of 3, where the calculation can easily be done by identifying that 9 is 3 bundles of 3 so the total cost would be 3 bundles times $1.20. This flexibility in thinking seems obvious to experienced adults but is often not considered by children who are more comfortable following a procedure or rule. Remember that reasoning disappears when students blindly apply rules.

**Teaching Implications:**

Teach students to look for the unit. Children do not readily recognize the reference point upon which the whole problem is built so often make inappropriate assumptions. Explicitly ask what the unit is? “For many students progress in fraction thinking is greatly delayed .all because they never grasped the importance of unit” (Lamon, 1999) Teachers should use units of different types, one continuous item such as one cake, more than one continuous item like 3 pizzas, one or more continuous objects that are perforated or partitioned such as a candy bar, discrete objects such as a group of marbles, discrete objects that are arranged in a certain way like tennis balls in
a can, composite units like 6 pieces of gum in a package. Ask students to give the answer to the same question in the context of several different units as illustrated in Example 3.3.1 and 3.3.2.

<table>
<thead>
<tr>
<th>Example 3.3.1 Different packs</th>
<th>Example 3.3.2 Different units</th>
</tr>
</thead>
<tbody>
<tr>
<td>You drank 30 soft drinks last month.</td>
<td>How much does ▲▲▲▲▲▲ represent if…..</td>
</tr>
<tr>
<td>How many 6 packs is that?</td>
<td>▲▲ is the unit?</td>
</tr>
<tr>
<td>How many 12 packs?</td>
<td>▲▲▲ is the unit?</td>
</tr>
<tr>
<td>How many 24 packs?</td>
<td>▲▲▲▲▲ is the unit?</td>
</tr>
</tbody>
</table>

Allocate time for part/whole reasoning before algorithmic operations so that students understand and can explain problems such as 2/3 of 12 marbles is 8 marbles because they can think of the 12 single marbles as three 4 packs (unitizing).

Help develop reasoning abilities rather than following a unit price approach by encouraging students to unitize into efficient chunks. Have students determine multiple solution methods and compare and contrast them for convenience. Students need to reason up and down, coordinating size and number of pieces, a mental process useful in the development of proportional reasoning. Students should be encouraged to look for multiple ways to solve and then compare which way is more efficient. They should be given credit for each different way they can come to the correct answer. In Example 3.3.3 different chunks which can be used to solve this problem. Comparisons of the unit price to chunking into 2 oz. or 4 oz. or 8 oz. prices could be made. Students should be asked: Which chunk is easier to do in your head? Would it be easier to scale up instead of down?

<table>
<thead>
<tr>
<th>Example 3.3.3 Which cereal is a better buy?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choco berries 16 oz. for $3.36 or Choco berries 12 oz. for $2.64?</td>
</tr>
</tbody>
</table>
3.4 PARTITIONING

Partitioning is the process of dividing an object or objects into a number of disjoint and exhaustive parts. This means that the parts are not overlapping and that everything is included in one of the parts. Partitioning lies at the very heart of rational number understanding. Fractions and decimals are both formed by partitioning. Decimals are based on a division of a unit into 10 equal parts and each of those parts into 10 equal parts. Locating a rational number on the number line depends on the division of the unit into equivalent spaces. The roots of understanding equivalence are laid when performing different partitions that result in the same relative amount; partitioning is an outgrowth of fair sharing. Early on students cut every whole into the same number of pieces, but as partitioning skills develop students look for the most efficient sharing. For example, if 4 people are going to share 5 pizzas, students will start by cutting each pizza into 4 pieces and giving each person one piece from each pizza. Later they will recognize the more efficient method of giving each person one whole pizza and then partitioning the remaining pizza into four pieces for distribution. Partitioning is essential to understanding the important question related to quotients. How much is one share? This is also the foundational understanding for unit rates, how much change in one quantity relative to a change of one unit in another. Students must understand the relationships among critical quantities, the number of people, one share and the unit.

There are two types of partitioning: partitive and quotitive division. Partitive is the equal sharing idea discussed above. Quotitive is how many shares can be measured out of some larger quantity? In this type of question the divisor becomes the new unit. When you buy a slice of pie at a restaurant suppose you get 1/3 of a pie, if they have 4 ½ pies left how many slices can be served? The answer is 13 ½ slices and slices are an entirely different unit than what we started with. The ½ is not ½ of a pie but ½ of a slice or 1/6th of the pie.
“There are many middle school students who cannot understand these quotient and part-whole questions because they need more partitioning experiences.” (Lamon, 1999)

**Teaching Implications:** Partitioning is a concrete activity and doing promotes insight. Asking the right questions to develop student understanding is essential. Ask how much is in each share and require students to identify the unit. Ask what part of the original amount one share is.

---

Example 3.4.1 Identify the share and the unit

1. If 5 people share 4 pancakes, how much will each person eat? 
   How much of the pancakes will each person eat?
2. If 4 people share 2 (6-packs) of lemonade, how much will one share be? 
   What part of the unit is one share?

---

In quotitive division, ask how many shares there are. Help students understand the differences between how much in one share and how many shares can be made.

---

Example 3.4.2 How many shares?

1. If each person needs a piece of rope 5/8 yard long, how many people can get a piece from 6 yards of rope?
2. If each person needs 8 oz. of lemonade to drink, how many people can drink from 1.5 gallons of lemonade?
3. If you need 3/5 of a foot of ribbon to make a table decoration. How many table decorations can you make with 7 feet of ribbon?

---

Elicit different partitions and encourage students to look for equivalent fractions. Partition different kinds of units, continuous, discrete, arrays, composite units, different shaped items, etc. Do not always use nice or familiar fractions. Rational numbers as measures are good places to practice partitioning on a continuous scale. See Example 3.4.3.
3.4.3 Partitioning measurements

1. Find three fractions between $\frac{4}{7}$ and $\frac{5}{7}$.
2. $4.5$ is half way between $4$ and $5$. So is $\frac{4.5}{7}$ half way between $\frac{4}{7}$ and $\frac{5}{7}$? Explain.
3. How can $\frac{4.5}{7}$ be written as a rational number? Explain.

3.5 RATIONAL NUMBER INTERPRETATIONS

Rational numbers build on student understanding of fractions but are not equivalent to fractions. The word fraction is fraught with ambiguity and various interpretations both within and outside the math community. Fraction can be used as an expression for a small amount, a symbolic form for writing numbers, part-whole relationships or a rational number. Fractions themselves present a big mathematical and psychological stumbling block to students. Just as multiplicative reasoning is a big leap from additive thinking, fractions requires students to work with new models and methods for adding, subtracting, multiplying and dividing.

Understanding rational numbers involves understanding that there are many different meanings that end up looking alike when they are written in fractional symbols. Unfortunately because of the difficulty in helping students comprehend fractions, instruction has traditionally focused on only the part/whole comparisons. This emphasis has led to confusion about the other meanings inherent in rational numbers that are essential for proportional reasoning. As Lobato (2010) points out fractions and ratios have overlap but do not have the same meanings. It is important that teachers and students understand these distinctions. Proficiency isn’t just about manipulating the fractional symbols but looking at the context for interpretations Kier (1993) identifies five major interpretations of rational numbers. The fractional symbol of $\frac{3}{4}$ has different meaning in all five contexts: refer to Example 3.5.1
Example 3.5.1 Five rational number contexts.

1. Part-whole
   Home in 45 minutes \(\frac{3}{4}\) of an hour
   Three quarters in my pocket \(\frac{3}{4}\) of a dollar
   The chance of drawing a blue sock from 12 blue and 4 red socks \(\frac{3}{4}\) chance

2. Operator
   The picture was \(\frac{3}{4}\) the original size. \(\frac{3}{4}\) in size
   I want 3 copies of \(\frac{1}{4}\) of that \(\frac{3}{4}\) copies
   Tokens are 3 for 4 quarters. Put in 12 quarters \(\frac{3}{4}\) the number of quarters in tokens

3. Quotient
   Three different cookies cut in fourths eat one piece from each \(\frac{3}{4}\) pieces eaten
   A package of 3 cupcakes shared with 4 people \(\frac{3}{4}\) cupcake per share
   How many \(\frac{1}{2}\) portions can I make from \(\frac{3}{8}\) of a pie? \(\frac{3}{4}\) portions

4. Measure
   The point \(\frac{3}{4}\) on a number line \((a\ distinct\ point)\)

5. Ratios
   A turtle moves 6 inches in 8 minutes \(\frac{3}{4}\) inch per minute
   There are 12 men and \(\frac{3}{4}\) as many women \(The\ women\ are\ \frac{3}{4}\ the\ number\ of\ men)\)
   There are 3 blue socks for every 4 men. \(3\ socks\ to\ 4\ men\)

These 5 interpretations illustrate numerous contexts: percent, probability, ratios, rates, the number line, fair sharing, exchanging, sampling, scaling, etc... A closer look at each interpretation highlights some of the issues that may need to be addressed to facilitate student understanding of ratios, rates and proportions.

**AS PART/WHOLE COMPARISONS**

Students struggle with the meanings of part-whole fractions and the early ideas of equivalency and comparisons because they continue to inappropriately apply whole number ideas to fractional operations. Teachers spend large amounts of time working within this model and therefore a part-whole comparison is how rational numbers are most commonly perceived. In this interpretation a rational number is used to compare one or more equal portions of a unit to the total number of like-sized equal portions in the entire unit. There are several subtle implications that are being communicated when rational numbers are only taught as part-whole...
comparisons. The numbers form an ordered pair, the numerator or top number always represents the number of pieces you have, while the denominator or bottom number always represents the total number of equal pieces in the unit. The number of portions and sizes of the portion depend on the way the unit is unitized but are always unitized in the same way. Although a different unitization results in a different fractional name the quantity remains the same. Traditionally these concepts are often taught without a proper emphasis on the unit. Teachers routinely use the same pizza or cake for every example or comparison. This type of instruction leads students to become rigid in their conceptualization of the representation of a fraction: always the same, the same top and bottom units as well as visualizing the same partitioned whole. The proportional nature of part-whole fractions is de-emphasized when this sameness is employed.

Part-whole understanding is important for comparisons purposes. Students who are taught to reason about the number of and size of pieces when pictures are not practical begin to develop important skills necessary for proportional reasoning. Reasoning allows students to visualize and unitize without relying on integer results: 8 eggs/12 eggs is then equivalent to \(1\frac{1}{3}\) (6 packs) to 2 (6 packs) or 2/3 dozen/ 1 dozen. This kind of flexibility requires time spent on part-whole comparisons before algorithmic fractional operations are taught.

**AS OPERATORS**

Rational numbers act as operators in the context of mapping or scaling. This notion is about shrinking and enlarging, contracting and expanding, enlarging and reducing, multiplying and dividing. As operators, rational numbers transform line segments, number of items and figures in geometry. Operators make similar figures of different proportions. This operator role is very different from the part-whole representation. The operator “3/5 of“ defines a composition of multiplication (making something 3 times larger) with division (making a reduction by 1/5)
The operator interpretation lends itself to a machine representation with inputs and outputs. In the example of an input of 4 quarters with an output of 3 tokens, students can see an input of 12 quarters will output 9 tokens. Understanding rational numbers as operators means (1) students can interpret a fractional multiplier in a variety of ways such as three \( \frac{3}{4} \) units or \( \frac{3}{4} \) of a 3 unit; (2) students can combine two operations one right after the other such as multiply by 3 and then divide by 4 or divide by 4 and then multiply by 3 into a single fraction named \( \frac{3}{4} \); and (3) students can identify the operator by observing the input and output. All three of these skills use multiplicative thinking and greatly facilitate proportional reasoning.

**AS QUOTIENTS**

A rational number is viewed as a quotient when it is the result of division. Understanding rational numbers as quotients is based on understanding the partitioning questions of; “how much is one share?” “how much does one receive?” and “how many shares can be made?” All these questions differ from the other interpretations of rational numbers and make no sense in the limited understanding of fractions as a part/whole representation. The answer to “how many \( \frac{1}{2} \) portions can I make from \( \frac{3}{8} \) of a pie?” is \( \frac{3}{4} \) portions. This number makes no sense when interpreted as 3 out of 4 since it clearly not \( \frac{3}{4} \) of the pie. There is not a commonality between the share chosen and the wholeness of the pie.

Rational numbers as quotients as well as rational numbers as ratios represent a unique challenge to students because they must not only attend to numerical calculations but quantitative processes as well. The relationship between the divider and divisor must be conceptualized to understand ratios, rates and proportions. Unit rates rely on a student’s ability to visualize a fractional answer not as just a numerical representation but as a contextual situation as well.
AS MEASURES

When we talk about rational numbers as measures, the focus is on successively partitioning the unit until a perfect fit is obtained. Partitioning in this sense says instead of comparing the number of equal parts you have to a fixed number, the number of equal parts in the unit can vary and what you name your fractional amount depends on how many times you are willing to keep up the partitioning process. All rational numbers can be partitioned while irrational numbers cannot. When students order rational numbers, they are using the interpretation of rational numbers as a measure of the distance from zero and placing them in their respective locations. Although this interpretation seem straight forward students continue to struggle long past their middle school years in correctly placing unfamiliar rational numbers on the number line. This speaks to the importance of devoting sufficient time to partitioning exercises.

AS RATIOS

Ratios are not always rational numbers but many rational numbers represent ratios. Ratios use quantities not just numbers and this additional requirement is where much difficulty lies for students and teachers. The differences between fractions and ratios are often blurred. A study by Clark et al. (2003) undertook investigating how using the subcontracts of rational numbers especially ratios helps students make connections. They looked at how the use of ratios and fractions by teachers and textbooks affected student learning. As they examined the way teachers and textbooks viewed the interconnection between fractions and ratios they discovered that in most cases ratios and fractions are taught as topics either unconnected or connected only as alternatives in notation.

“The introduction of fractions as a notation for ratios, disconnected from students’ prior experience with fractions, seems to inhibit their problem-solving abilities. The relationship between ratios and fractions is ambiguous, when we consider how teachers and textbook authors present it, but one finding in unambiguous: The middleschool students in our study, even those who perform well in school and on our proportional-reasoning test, are not making the conceptual connections between ratios and fractions.” (Clark, 2003)
A number of mathematical connections exist between fractions and ratios. Ratios are often expressed in fraction notation to give meaningful interpretations but do not have identical meanings. Ratios and fractions can be thought of as overlapping sets. The biggest distinction is that ratios often represent part-part or related set comparisons while fractions do not. For example, the ratio $\frac{3}{4}$ can be 3 girls to every 4 boys in a classroom of 28 students; it cannot be understood as a fraction representing 3 girls out of 4 boys. Mentally constructing ratios requires students to use the surrounding words as contextual support to facilitate the transition from ratio as a mere number (or fraction which it is not) to a situation (or relationship between quantities). Ratios should not stand alone as a number but include the relationship they represent. Teachers as well as students need to realize the importance of using labels while doing computations with ratios “this will keep the difference between ratios and part/whole fractions from becoming fuzzy.” (Lamon, 1999)

**Teaching Implications:**

Instruction is greatly influenced by a teacher’s understanding of rational numbers. An emphasis on only one interpretation handicaps students’ subsequent mathematical progression. It is vital to highlight the sub constructs of rational numbers and explicitly teach students to see the differences between part/whole, operators, quotients, measures and ratio interpretations (refer to Example 3.5.1).

Helping students transition from a part-whole understanding to other interpretations facilitates the multiplicative thinking necessary for proportional reasoning. Comparing and contrasting meanings facilitates the flexibility needed in student thinking to increase rational number comprehension.

Students’ work should be analyzed to determine within which construct they are manipulating fractional representations. By requiring students to contextually label
representations teachers can help students understand the ways rational numbers can be used to represent proportional relationships.

3.6 RATIO SENSE

Developing ratio sense is imperative in proportional reasoning because the precise definitions and meanings of ratios and rates come from the context of the problem. Part of the problem for instructional purposes is that the everyday language and usage of rates and ratios is often inconsistent, less than correct and difficult to precisely define. The common core states that a ratio associates two or more quantities and that ratios have associated rates. The emphasis is on a ratio’s ability to describe relationship between quantities. To develop ratio sense teachers need to build on the fractional representation as a way to order and judge equivalence of ratios while introducing the context implicit in the use of ratios. As students begin to examine the relationships ratios represent they can then determine the ways in which ratios can be used to problem solve. Four different types of ratios are important in proportional problems: part-part-whole, associated sets, well-known measures and growth (stretching and shrinking) (Langral, 2000)

The first part of ratio sense is the distinct difference between part-whole comparisons and part/part constructs. Discrete objects such as black and white chips make this obvious: 2 black chips out of 3 total chips are distinctly different from the 2 black to 1 white chip comparisons. Students have intuitively been making these types of comparisons since early elementary ages and can see the difference. This understanding is kept intact when models or manipulatives are continually used to remind the students that they are working with relationships rather than just a number. Students progress from these more intuitive part-part comparisons to associated sets, well-known measures and growth ratios. As students become familiar with models they can
move to using labels to represent the quantities relationships. The emphasis must always be on describing the relationship rather than finding a number.

Looking at the contexts of ratios for extendibility, reducibility, reversibility, homogeneity and divisibility gives understanding to the relationship ratios represent. Ratios can be extendible but not always. Extendibility means the ratio can be used to represent a proportional relationship and allows predictions to be made. A comparison of your current age to someone else’s age is an example of a ratio that is not extendible. The value of the ratio will change in just one year. This lack of extendibility means it cannot be used as a predictor. Many ratios however have their value in the extendibility which allows them to be used as predictors: the ratio of blue to red paint to make purple paint, the ratio of mechanical failures to the number of machines produced, the number of cups of flours to cups of sugar in a recipe, etc. Looking at the context for extendibility is important to highlight before students create various equivalent ratios.

Reducibility of a ratio is used to help us understand the situation; it reduces the ratio to an easily visualized multiple. Although the number of M&M’s in the package may be seen as 5 blue to 55 brown, it is meaningful to reduce the ratio to 1 blue to every 11 brown. We lose the information of how many M&Ms are in the current package but we gain the ability to use this ratio as a predictor of the number of blue M&M’s given the number of brown. This example also illustrates the idea of reversibility. Reversing a ratio is not like taking the inverse of a rational number although numerically it appears equivalent. We cannot invert the ratio and return a quantity to its original size but we can reverse the ratio (flip the quantities) to describe the relationship from a different perspective. We can say that for every 11 brown M&M’s there will be 1 blue. This type of dual nature of ratios is an important distinction to explicitly make; the reciprocal of a fraction and the reversibility of a ratio are entirely different constructs.
The homogeneity of a ratio relates ratios to measures. If a ratio is homogeneous we can break down the ratio into any size partition we desire and the ratio will remain the same. The percent of alcohol in water, density measures, constant mechanical rates, gas mixtures and etc. are all examples of homogeneous ratios. Discrete objects do not have this same homogeneity. We cannot really take 3 girls to every 4 boys and say that we will have 1.5 girls for every 2 boys. Although this reduced ratio may be useful as a predictor, it is not homogeneous; we cannot actually partition 1.5 girls to 2 boys.

Ratios as quotients use the divisibility of a ratio to create indices and unit rates. Well-known measures such as speed, slope, unit price or density are not always recognized by students as being a ratio because the divided quantity is perceived as a single entity. In addition some types of ratios serve as descriptions of fixed relationships. They may describe physical properties such as π which describes the ratio of the circumference of a circle to its diameter or the number of calories in one apple. Divided ratios obscure the input information but highlight the homogeneity and extendibility of ratios. Students need to be given experiences in these well-known measures so they can understand how change and invariance work together. They need to be able to identify the ratios being represented by the indices. Developing understanding of ratios as measures requires teachers to provide meaningful contextual experiences with these quantities. Lobato (2002) illustrates this important part of sense making in the context of slope. Rather than beginning instruction with a slope formula and definition, start with situations and poses measuring questions (refer to Example 3.6.1). This type of instruction allows sense making to be the primary goal rather than calculating slopes. As students identify the attribute, recognize the need to measure, and identify the quantities that affect the attribute, they come to understand the important information ratios can give to our understanding of the world.
Example 3.6.1 Developing an understanding of slope

<table>
<thead>
<tr>
<th>STEP ONE</th>
<th>Isolate the attribute that is being measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>Draw two non-identical ramps with the same steepness.</td>
<td></td>
</tr>
<tr>
<td>How do you measure the steepness of wheelchair ramp?</td>
<td></td>
</tr>
<tr>
<td>Is a ramp the same steepness at the top and bottom?</td>
<td></td>
</tr>
<tr>
<td>Given an example of two non-identical ramps with the same steepness, and have students identify what is alike and what is different.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STEP TWO:</th>
<th>Determine which quantities affect the attribute and how</th>
</tr>
</thead>
<tbody>
<tr>
<td>Have students use a computer model of a wheelchair ramp to change various characteristics such as length of the ramp, height of the ramp, direction and length of the platform individually. Have them determine which quantities affect the steepness</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STEP THREE:</th>
<th>Understand the characteristics of a measure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Have students test direct measures they suggest to see if they can find something that assures the steepness would be the same.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STEP FOUR:</th>
<th>Construct a ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Have students use two identified quantities as a ratio to create a measure of steepness.</td>
<td></td>
</tr>
</tbody>
</table>


When the distinctions between fractions and the fractional representation of ratios are blurred, students are often confused about ratios as operators. Ratios multiply and divide like fractions but add differently. Ratios add numerator to numerator and denominator to denominator something students have been taught to avoid like the plague. Helping students understand the context of ratios may actually help them discriminate these different operational methods. Teachers might enhance student understanding by correcting their addition mistakes of part-whole fraction by pointing out this works when we are taking about part-part comparisons. This would help students understand that rational numbers always have a construct.

**Teaching Implications:**

Ratio sense is an intuition acquired through experience in appropriate contexts. Students need to be given the opportunity to determine when using ratio is an appropriate problem solving technique rather than just identifying that a ratio exists.
Example 3.6.2 Ratio sense making

Discuss the statements below. Do they make sense? What distinguishes those that make sense from those that do not?

1. If one girl can walk to school in 10 minutes, two girls can walk to school in 20 minutes.
2. If one box of cereal costs $2.80, two boxes of cereal cost $5.60
3. If one boy makes one model car in 2 hours, then he can make three models in 6 hours.
4. If Huck can paint the fence in 2 days, then Huck, Tom and a third boy can paint the fence in 6 days.
5. If one girl has 2 cats, then four girls have 8 cats.
6. If a car travels 50 miles in one hour, it can travel 200 miles in four hours.

Each example in Example 3.6.2 has a ratio but the use of that ratio does not make sense in many of the contexts. By giving students problem solving with diverse solution methods, ratio sense is enhanced as students evaluate how ratios differ in the information they provide.

Teachers need to help students explicitly identify the differences between ratios and plain fractions. Help students identify the difference between adding ratios and fractions.

Example 3.6.3 Highlighting the contextual differences between ratios and fractions.

1. List as many ways as you can that 3 girls: 5 boys is not the same as 3/5.
2. In what ways are they the same?
3. How does the ratio 3 girls to 5 boys change if we add 1 girl and 3 boys?
4. How is that different from \( \frac{3}{5} + \frac{1}{3} \) ?

Teachers should emphasize the relationships represented by ratios rather than the numbers. Have students always use words and labels when working with ratios. Show students that by keeping the context in mind problem solving becomes easier. Example 3.6.4 illustrates how the context of a recipe encourages students to think about the problem before thinking about what numerical method to use.
Example 3.6.4 Using context to enhance ratio sense.

To make beef sandwiches for 3 people, Jose used a half pound of roast beef. How much roast beef does he need to make 17 sandwiches?

Teachers should help students look for the extendible, reducible, reversible, homogeneous and divisible of the context to highlight the nature of ratios. This can be accomplished by explicitly teaching students to recognize the descriptive nature, predictive power and dual nature of ratios. Ask students to create situations where a ratio could be useful to make a prediction. Have students demonstrate ways to utilize the dual nature of ratios.

Example 3.6.5 Identifying the usefulness of ratios

1. What does miles per hour tell us about a situation?
2. If we have a constant ratio of 10 miles per hour, what can we predict?
3. When might we want to know we are going $\frac{1}{10}$ hours per mile?

3.7 ATTENDING TO QUANTITIES AND CHANGE

Most of a student’s knowledge of change is intuitive and built on personal experience. Prior to the study of change in algebra students need to develop a language for describing and talking about change, a way to categorize change and develop representations for the types of change they encounter. Proportional reasoning requires that students can evaluate and compare the relationship between two quantities and how they change. NCTM identifies the essential understanding required about change in these ways: (1) Reasoning with ratios involves attending to and coordinating two quantities, (2) Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest and (3) A proportion is a relationship of equality
between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.

How do students begin to recognize and coordinate what is changing? In general students of middle school age tend to focus on the physical properties of objects and situations rather than their measurable quantities. Consider a simple scenario: You ride your bicycle down the street. What changes? Typical answers include: I moved from where I started; my pedals went up and down; my wheels went around. These are physical properties rather than something measured. Here are examples of answers that could be measured. I went a certain distance, an amount of time elapsed, by using distance and time I could calculate my speed, my tires rotated a specific number of times, etc. Middle school students must be encouraged to look at change in quantifiable ways. Time as an element in change should be explicitly taught. Most students although exposed to the idea of speed as the distance divided by time do not isolate these attributes in evaluating change and recognize they can measure speed. Many quantities that we talk about as a single entity (such as speed, density, mpg, etc.) are not readily recognized by students as being two different changing attributes.

In one study (Lobato, 2010) when students were asked to create a way to measure steepness, they did not transfer their recent knowledge of slope to the situation because they did not recognize steepness or slope as the comparison of two measurable quantities. Because rates remain constant students do not understand that two quantities are actually changing and that it is the multiplicative relationship between the two that is invariant. Change as a focal discussion in the classroom is essential in helping students identify the different types of changes such as absolute, relative, average, rate of change, etc.

Another stumbling block to understanding quantities and change is in how students view the use of a variable. They often see a variable as an unknown static quantity rather than a
quantity that changes. Students need to understand that relationships that exist between dependent and independent variables highlight a constant relationship between the change of one variable and the change of another. This idea that a proportional relationship represents change and invariance in the same situation is an essential understanding.

An understanding of change also includes important discussions of the direction of change, the shape of change and the rate of change. Students should be taught to use arrow notation in discussing change so that their focus is on the relationship rather than the numerical quantities. By focusing on the change, students develop an intuition about evaluating their final answers. Example 3.7.1 illustrates this simple step that gives students a quick check of their final answers. If in question 1 a student incorrectly uses a proportion the time will increase instead of decrease. The arrow notation will identify the student error.

Example 3.7.1 Using arrow notation to think about change

1. 8 people painting a room take 5 hours, how long will it take 10 people to paint a room?  
   Number of people ↑ time ↓
2. For a party of 13 people you need 5 pounds of candy. If you want a party of 20 how much candy do you need?  
   Number of people ↑ amount of candy ↑

Discussion about the shape of change should also precede formal graphing techniques. Students can describe the shape of change with pictures (such as the changing phases of the moon) or with descriptive words such as abrupt, gradual, irregular, periodic or cyclic, always increasing, disappearing into nothing, changing direction, etc. Having students describe change helps prepare them to think about how to measure change. Often students are asked to think about “rate of change” when they have no prior change schema. Helping middle school students think about change as how fast one quantity is changing in relation to another quantity gives rate of change a context. The idea then that proportional relationships have a constant rate of change
then fits into the whole schema of change. Students can sketch graphs of two quantities charting the way the change of one affects the change of the other without using any numbers. Lamon (1999) illustrates how to have students sketch change with Example 3.7.2: The faster you run, the less time it will take to get there (↑↓).

Example 3.7 (b) Sketching change: The faster you run, the less time it will take to get there.

1) Decide which variable depends on which. Which value do you have to know before you can decide the other? That one is called the independent variable. Label the axes. In our example time depends on running speed.
2) Pick a reference point so that you can determine what more or less time means.
3) From the reference think about what happens to time as speed varies. When you run faster the time is shorter. When you run slower, the time is longer. Put some points on the graph
4) Think about the extremes so that the shape of you graph approaches these and sketch based on the three points.

This exercise forces students to think about change without any use of numbers and illustrates how important conversations about change are in helping students develop reasoning.

**Teaching Implications:**

Students need to discuss all types of change situations so that eventually they can recognize proportional relationship. In discussing changes, research has shown (Lamon, 1999) that teachers need to ask the following questions whenever students are solving problems: (1) Which quantities are changing? (2) Does the change in one quantity depend on the change in the other? (3) Are there any relationships that remain unchanging? (4) If two quantities are changing, in which directions are they changing? (5) Can you represent the change using arrow notation? (6) Is one quantity changing more quickly than another? How can you tell? and (7) Can you sketch the relationship between the two changing quantities.
In addition students need to be given tasks that only require them to think about and identify changes without using any numbers.

Example 3.7.3  Describe what changes and what stays the same

1. You are draining the water in your bathtub.
2. You are filling your car’s gas tank.
3. You walk to school.
4. You make an enlargement of a picture.
5. You stretch elastic.

Students should also be asked to evaluate statement of change as in Example 3.7.4

Example 3.7.4  Evaluating statements of change

Some of the statements given below are correct; others are not. Use arrow notation to think about the relationships in the statement. If the statement is incorrect (does not make sense) change one number so it is correct. If you can’t correct the statement explain why.

1. If 1 bag of salt weighs 40 pounds, 3 bags should weigh 120 pounds.
2. If it takes 6 people 1 hour to clean a house, then it would probably take 3 people about 1 \( \frac{1}{2} \) hours to clean it.
3. If 1 boy has 3 sisters than 2 boys probably have six sisters.

Lamon (1999)

4. MCF RESEARCH FINDINGS

The theory of conceptual fields provides a framework that includes complementary foundational mathematical elements from different perspectives. Socioconstructivism is acknowledged as contributing to an understanding of how children acquire mathematical concepts but other perspectives, namely intuitionism and formalism, are acknowledged for the contributions that they make to the development of mathematical constructs. Mathematics, its concepts and theorems are the essence of the theory of conceptual fields. In addition, the theory, being underpinned by the work of Piaget and Vygotsky, includes the psychological constructs necessary to understand cognitive structures. The theory also provides the tools for analysis of
problem situations and learner responses. Vergnaud (1994) stresses the importance of analyzing both mathematical situations and learner schemas in terms of mathematical constructs, so that the path of mathematical development may be identified and therefore guide instruction. Much of the work previously summarized is from MCF research findings. In addition, several articles of additional importance are worth elaborating in our attempt to identify key factors in developing proportional reasoning.

The multiplicative conceptual field research makes the observation that how students understand a concept has important implications for what they can do and learn subsequently. While this is not a new observation it means that the design of curricula and teaching requires careful analysis of not just what we expect students to learn but what students actually learn from instruction. It means that the way beginning concepts such as multiplication and fractions are taught have large implications on the schema students develop. Thompson (2003) makes a case for the implications fraction understanding has on multiplicative reasoning. For example, many students understand “a/b” as denoting a part-whole relationship, that “3/7” means “three out of seven. This becomes problematic when they attempt to interpret “7/3” because you can’t have 7 out of 3 things.

One problem may be the way teachers themselves view fractions, and whether or not they understand how their understanding of fractions influences their understanding of ratios and rates. Post et al. (1991) tested late elementary teachers.

“Our results indicate that a multilevel problem exists. The first and primary one is the fact that many teachers simply do not know enough mathematics. The second is that only a minority of those teachers who are able to solve these (proportions) problems correctly were able to explain their solutions in a pedagogically acceptable manner.”

Thompson notes these results not as a criticism of current teachers but to highlight the need for dialogue on the important understandings that are essential in developing multiplicative
thinking skills. He then goes on to describe the controversy that exists in defining rational numbers,

”We wish to emphasize that the mathematical developments of rational and real number systems interconnect many issues that typically are not treated until advanced undergraduate or introductory graduate mathematics courses. As such, we have no idea what school mathematics textbook authors or other writers intend when they say they want middle-school students to “understand” rational and irrational numbers.”

To understand entails considering numerous mental constructs (percepts, images, concepts, thoughts, words, etc.) as well as result of operations or rather operating knowledge of a subject. It is the way people use their schema to reason through problem solving through fractions that indicates their understanding. Thompson’s stance is that coherent fractional reasoning develops by inter-relating several conceptual schemas often not associated with fractions such as multiplicative reasoning.

How the measurement of quantities is perceived is one such construct. To conceive of a measured quantity is to imagine the measured attribute as segmented. At the heart of measurement is a ratio. A quantity’s magnitude (the amount of stuff) is independent of the unit in which it is measured. A change of unit in measurement does not change the quantity’s magnitude. If we measure it in units of 1 or in units of ¼, the quantity’s magnitude remains unchanged. Students often miss this important distinction between a quantity and its numerical measurement. This distinction is apparent in quantity equations such as area or volume where the numerical calculation can yield several “different” numerical values depending upon the measurement but the quantity’s magnitude obviously remains unchanged. There are two ways then to conceptualize measurement: Measure as a number of things versus measure as a ratio comparison: 12 little lengths called inches or the total length of “this” is 12 times as large as the length of “one inch”. Understanding measurement as a ratio comparison highlights the way multiplication must be conceived in relation to fractions.
It is important to keep a distinction in our minds between conceptualizing, and the activities of multiplying and dividing. Generally, students do not see proportionality in multiplication. In fact, a large amount of curriculum and instruction has students understand multiplication as a process of repeated addition. However, extensive research has shown how repeated addition concepts become limiting and problematic for students when fractions are introduced. Repeated addition is a way to quantify a number rather than the concept. Multiplication should instead be conceptualized as a multiple proportion...making identical copies of some quantity. The key is to get students to think about the quantity before they think about the numerical calculation. It is how students decode a mathematical statement that has large impacts on the connections they form. If they read 5 X 4 as five fours then 5 2/3 X 4 means five and two-thirds fours. This tells them to use proportions rather than repeated addition. It means that the answer will be 5 and 2/3 times bigger than the original number four.

“We re-emphasize that when a curriculum starts with the idea the “___X___” means “some number of (or fractions of) some amount,” it is not starting with the idea that “times” means to calculate. It is starting with the idea that “times” means to envision something in a particular way---to think of copies (including parts of copies) of some amount. This is not to suggest that multiplication should not be about calculating. Rather, calculating is just one thing one might do when think of a product. Non-calculation ways to think of products will be important in comprehending situations in which multiplicative calculations might be useful. The comprehension will enable students to decide on appropriate actions.”

With this understanding, students can conceptualize fractions as entailing a proportion. If students’ image of fractions is “so many out of so many” it implies that they have something in common so when we ask a question such as “what fraction of the girls is the number of boys?” students are puzzled and give up understanding. To think multiplicative means that a fraction say 4/5 means 4(1/5) or four copies of one fifth. It entails understanding that the new quantity is 4 times bigger than 1/5, or that the new quantity is 1/5 as large as 4. When presenting this concept in professional development settings, many teachers argue that repeated addition is so much easier. Additive reasoning is easier than multiplicative reasoning but is problematic for
mathematical growth. A multiplicative fraction schema is based on conceiving two quantities as being in a reciprocal relationship of relative size: Amount A is 1/n the size of amount B means that amount B is n times as large as amount A. Or in other words the two amounts in comparison are each measured in units of the other. Amount B being 7 times as large as amount A is to say that amount B is measured in units of A. Students gain considerable mathematical power when they utilize a schema of operations that relies on reciprocal relationships of relative size. This type of understanding is extremely rare in the U.S. but prevalent in other countries.

Thompson (1994) asserts that student misconceptions may be predominantly the effects of schooling:

“when we remove certain insidious practices of schooling, such as insisting that students be able to calculate any expression they encounter in solving a problem, and focus students’ attention on their conceptions of situations, students can advance far beyond what we typically expect of them”

His premise is that people reason about situations that involve things and relationships. His project aimed to capture the multiple reconstitutions that take place as students’ progress toward the construction of mathematical objects such as ratio and rate and in particular speed. Quantitative reasoning is at the heart of this construction. Thompson clarifies what he means by quantity and a quantitative operation. His distinctions draw heavily on Piaget’s notions of internalization, interiorization, mental operation and schema. Quantities are conceptual entities. They exist in people’s conceptions of situations and entail some form of measurability. A quantity is schematic and is composed of an object, a quality of the object, an appropriate unit or dimension and a process by which to assign a numerical value to the quality. Quantification is a process by which one assigns numerical values to qualities and is a process of direct or indirect measurements. The only prerequisite for a conception of quantity is to have a process in mind.
A quantitative operation (others refer to this as qualitative reasoning) is a mental operation by which one conceives a new quantity in relation to one or more already conceived quantities. For instance, comparing two quantities additively creates a difference; comparing two quantities multiplicatively creates a ratio. A quantitative operation is different from evaluating the constituted quantity. A quantitative operation creates a structure—the created quantity in relation to the quantities operated upon to make it. The quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing. One interiorizes action, making mental operations imbued with the ability to comprehend situations representationally and then make inferences about numerical relationships that are not present in the situation itself. There is a clear distinction between quantitative operations that create quantities and numerical operations used to evaluate quantities. This distinction is largely tacit in mathematics education and the two are often confounded.

Students want to begin the numerical operations without evaluating the quantitative operations necessary to conceptualize the situation. Quantitative operations are non-numerical and have to do with comprehension of the system. In mathematical adults these two operations happen seamlessly and are almost indistinguishable. The NCTM essential understandings highlights that quantitative operations are essential in helping students achieve proportional reasoning. The ability to attend to and coordinate two quantities, isolate real-world attributes and determine the effect of changing quantities is all part of quantitative operations.

In a study of student conceptions of speed and ratio, Thompson (1994) highlights the imperative nature of developing this separation between quantitative and numerical operations. Initially, his subject abstracted speed as a quantity that could be measured from distance and used time as a ratio. Until she constructed an adequate quantitative operation between distance and time, she relied heavily on guess and check methods with no understanding of how to solve
problems. She often began numerical calculations without any idea why they may or may not yield appropriate answers. Aided by a computer simulation in which to try her answers and guided questioning, she moved from the idea that distance existed but time did not, to realizing distance involved a required time from which speed could be calculated as a ratio.

Several general conclusions where made: First, the standard method for introducing speed in schools is ineffective. Teaching “speed is distance divided by time” assumes that students have already conceived of motion as involving two distinct quantities (distance and time) and that they will gain understanding from merely being told this relationship exists. Both of these assertions are wrong. Secondly, what a child practices in a mathematics classroom often has little to do with developing habits of reasoning. Students must be given situations that are propitious for generalizing assimilations, accommodations and reflection. They need to re-evaluate and experience rather than being told. The final conclusions were that reasoning cannot progress until students move away from the drive to make numerical calculations and move towards trying to understand the situation.

Studying informal student reasoning is important in developing best practices. Missing value proportional problems have served as a venue for proportional reasoning. These studies have suggested that although teaching ratio reasoning through formal cross-multiplication or algebraic manipulation is historically prevalent, it has not been effective and may actually be harmful to developing reasoning. (Kaput, 1994) Psychological research on learning and cognition says that patterns of reasoning developed prior to formal instruction have effects on student approaches to problems that outlast or intrude in powerful ways on formally taught approaches. Kaput (1994) documents how building on existing student strategies of coordinated build-up/build-down processes, abbreviated build-up/build-down processes using multiplication and division and unit factor approaches increased students proficiency on various levels of
missing value proportional reasoning problems significantly more than they traditional formal approaches. In addition students were able to make sense of their approaches and better articulate their methods.

The comparison between the informal reasoning patterns and corresponding formal equation highlights several distinctions: In the informal approaches, the computation is a natural extension of the conceptualization; students undertook quantitative reasoning prior to numerical calculations. In the formal, equations-based approach one acts upon the formal system of symbols and syntactical rules rather than any conception of the situation. The calculations were thus quantitatively devoid and the students had no sensible interpretation of the system being modeled. Rate-conception and an appreciation of its equivalence are at the heart of proportional reasoning. Analyzing the four different methods of student solutions revealed that the most primitive informal strategy (building up/down) did not include this reasoning but remained contextually based. The abbreviated strategy increased the level of understanding by using the principle of homogeneity. It was both contextually and computationally more efficient but still had a relative low demand on a rate conception when numbers were easily divisible. The unit rate strategy obviously relied on rate conception. The formal methods however seemed to be devoid of both quantitative analysis and rate conception. In reviewing the steps involved in building the equation students need 1) a gross distinction between quantities, 2) a relation between the total amount and the unit amount which could often be determined by simple size discrimination and 3) the preservation of this relationship between two quotients. This procedure could easily be followed with any quantitative understanding of the situation. Another conclusion of the study which categorized tasks as well as reasoning is that interspersing addition problems is necessary to detect whether children were really reasoning quantitatively about a situation or just depending on a set algorithms. They suggested that student reasoning may take
the form of type detection in order to apply an algorithm learned for a certain problem type rather than attending to the situations and reasoning through them. In many respects textbooks have led students away from situation reasoning rather than towards it. The problems are designed to facilitate numerical computation rather than quantitative reasoning as well. Curriculum needs to provide students exercise in discriminating between additive and multiplicative situations so that students look for both absolute and relative differences. 

Schema based approaches have suffered some concern because of their inflexibility but proportion reasoning research can be incorporated into schema based approaches. Jitendra (2011) proposed a method in which flexibility and SBI are combined in helping 7th graders comprehend and solve proportion problems involving ratios/rates, scale drawings, and percents. First students are taught to evaluate the context for the type of question or problem being asked by modeling the situation with a schematic. They looked to see if the relationship was proportional or not. If proportional they reviewed the type of similar problems they had solved. Second, they compared and contrasted multiple strategies they had used in the past: models, tables, equivalent ratios or a proportion. They then chose a preferred method for problem solving. The final step was a think-aloud where students described the why and what they were doing to solve the problem. This method was shown to be successful with all types of students. It incorporates the research from both special need students and MCF to incorporate flexibility and reasoning into a schema that helps students think proportionally.

5. DEVELOPING PROPORTIONAL REASONING THROUGH TASKS

The mathematical community has long known that proportional reasoning “was of such great importance that it merits whatever time and effort must be expended to assure its careful development” (NCTM, 1989). This paper has attempted to highlight many of the significant
pedagogical implications that decades of research have revealed about the concepts students must understand to develop proportional reasoning. Teachers must take this information and translate it into student tasks that will build proficiency. There are six major types of proportional task (See Figure 5.1) students should master in middle school.

Figure 5.1  Six Identified Types of Proportional Tasks

1. **Part-part-whole:** Mrs. Jones put her students into groups of 5. Each group had 3 girls. If she has 25 students, how many girls and how many boys does she have in class?
2. **Associated sets:** Elena, Jim and Steve bought 3 helium-filled balloons and paid $2 for all 3 balloons. They decided to go back to the store and buy enough balloons for everyone in the class. How much did they pay for 24 balloons?
3. **Well-known ratio measures:** Dr. Day drove 156 miles and used 6 gallons of gasoline. At this rate, can he drive 561 miles on a full tank of 21 gallons of gasoline?
4. **Growth (stretching and shrinking situations):** A 6”X8” photograph was enlarged so that the width changed from 8” to 12”. What is the height of the new photograph?
5. **Percent:** The Orlandos had $400 to spend on camping gear. If they spent 21% of their money on sleeping bags, how much did they spend on sleeping bags?
6. **Population Samples:** In a representative survey 9 out of 12 middle schoolers prefer doughnuts to candy bars. How many doughnuts would you predict you need for a group of 250 middle schoolers? How many candy bars?

Numerous curriculum, articles and on-line tasks are available for teacher use. The difficult question is how to make pedagogical choices about which tasks to utilize to build student competency. From my review of the research, I have developed four big ideas that will guide my own teaching of proportional tasks. (1) Conceptualize before computing. (2) Explicitly highlight multiplicative understanding. (3) Utilize diverse contexts, solutions methods and units. (4) Provide students opportunities to test their thinking.

All four of these ideas have at their center the importance of not only what tasks and questions students are asked to do but the mathematical discussions that are necessary to develop proportional reasoning. Implementing these big ideas in my teaching should help students
develop their own schema for working with proportional reasoning by encouraging them to engage in sense making.

5.1 CONCEPTUALIZE BEFORE COMPUTING

The research is clear that students are computationally driven. The drive to produce the right answer interferes with students developing reasoning. Emphasis must be placed on conceptualization prior to computation. If you increase the number of people attending, you must increase the amount of food you purchase. If the speed is faster, you cover the same distance in less time. Students need to be asked to answer questions based on quantitative reasoning alone prior to beginning computations. What happens to the speed if you double the distance and the time remains the same? How is the time required to paint a house affected if you halve the number of workers? How can you compare two different runners to see who is the fastest? Design an experiment to determine someone’s speed. These types of questions allow students to understand the context prior to working with numbers. Making sure that students must reason about contexts first before working with numbers allows them to begin sense making. Highlighting a full range of quantitative strategies such as drawing pictures, identifying the quantities involved, looking for relationships between quantities, identifying how things change, thinking about extremes, etc., allows students to understand the context prior to utilizing a shortcut algorithm. Example Task 5.1 Moon Weights illustrates how I will modify problems to help students conceptualize before computing a numerical answer.

Example Task 5.1 Moon Weight

Typical Textbook Question
Physics tells us that weights of objects on the moon are proportional to their weights on Earth. Suppose a 180 pound man weighs 30 pounds on the moon. What will a 60 pound boy weigh on the moon?
Revised Question

Physics tells us that weights of objects on the moon are proportional to their weights on Earth. Suppose a 180 pound man weighs 30 pounds on the moon. What happened to the man’s weight? How much did it change? Can you measure change in another way and get a different answer? What will a 60 pound boy weigh on the moon? Which type of change helped you make your prediction?

5.2 EXPLICITLY HIGHLIGHT MULTIPLICATIVE UNDERSTANDING

Explicitly highlighting differences between additive changes and multiplicative changes allows students to build a schema for understanding proportional relationships that builds on their existing knowledge. The idea of multiplication as copies is extremely useful in helping students understand measurement and fractions. Discussing rational numbers as operators means \( \frac{3}{4} \) is visualized as 3 copies of \( \frac{1}{4} \) of the original and implies 3 times bigger than \( \frac{1}{4} \). This explicit multiplicative framework makes a transition to relative thinking easier. Relative thinking is a cognitive skill built upon understanding that a multiplicative comparison to another quantity is being made for example \( \frac{1}{4} \) the size. In all 6 of the identified proportional problem tasks, making a multiplicative comparison is imperative to thinking proportionally. For example, to solve the associated set problem in Figure 5.1, a student could ask how many times bigger is 24 than 3? The answer to this question then could multiplicatively be applied to the price of the balloons. Students draw the logical conclusion that if the number of balloons is 8 times more, than the price of the balloons is 8 times more. This language of “times” as copies then makes schematic drawings, ratio tables or formal algebraic proportion representations fit the multiplicative comparison thinking of proportional reasoning.

Students need to be explicitly taught to see the multiplicative nature of proportional tasks by the questions and activities teachers use in the classroom. The usefulness of explicitly teaching percent as a multiplicative index can be illustrated with this example:
The Orlandos had $400 to spend on camping gear. If they spent 21% of their money on sleeping bags, how much did they spend on sleeping bags?

What does 21% mean?  It means that compared to the whole you only have 21/100 or 0.21 times as much. If students interpret percent this way, they no longer are puzzled about what operation to use. Intuitively they know from personal experience what it means for something to be bigger or smaller than the original. Scaling tells them to use multiplication to solve the problem. In contrast, if students interpret 21/100 as being merely 21 out of a hundred they are left needing to build the proportional link necessary to solve the problem. By explicitly teachings 21% as a multiplicative index (how much smaller or bigger than the whole) as opposed to just a number out of 100, students build a schematic of why a percent is useful.

Explicitly identifying the equivalency of ratios as a multiplicative comparison is important. If y/x is the constant 3, it means that y is always three times bigger than x, or if y/x is ¼ then y is always ¼ the size of x. A useful geometry task of look-alike rectangles (Example Task 5.2) illustrates how students can be lead to explicitly comprehend the multiplicative nature of equivalent ratios in similar rectangles.

Example Task 5.2  Look-Alike Rectangles

1. Provide groups of students with ten rectangles which are numbered and printed on graph paper with the following side ratios: three with a ratio of 3 to 4, three with a ratio of 1 to 3, three with a 2 to 1 and one with a ratio of 1 to 1. Ask the students to group the rectangles into 4 different groups that look alike.

2. When students have finished grouping the rectangles have them discuss the reasons they classified the rectangles as they did.

3. Next have the students measure and record the sides of each rectangle and compute the ratio of long side to short side.

4. Have students answer the questions “How are the ratios and groupings related? Should they be related?”

5. How does multiplication apply between similar rectangles?
5.3 UTILIZE DIVERSE CONTEXTS, SOLUTION METHODS AND UNITS

Unintentionally, teachers choose familiar contexts and units to work with students in a hope to build skill proficiency. Often they simplify numbers, provide only the needed information and even identify what type of techniques should be used to solve the problem. This consistency and simplification in both contexts and units allows children to bypass the important conceptualizations that will develop reasoning and sense makings, and instead rely on procedures or rules.

The need to compare and contrast is paramount to sense making. The ability to identify when proportional reasoning is useful depends on developing an ability to identify differences: between additive and multiplicative, between absolute and relative change, between change in quantities versus change in relationships, between proportional and not proportional, between measurement and description, between extendible and not, etc. Teachers need to choose tasks that help students engage in thinking about differences both within and between problems. Types of questions should be varied within similar contexts so that students can identify differences. Example Task 5.3 requires students to examine and explain three different types of changes and comparisons within a single context.

Example Task 5.3 Task: Weight Loss

Max, Moe and Minnie are each on a diet and have recorded their weight at the start of their diet and at two-week intervals. After four weeks, which person is the most successful dieter based on the table below?

<table>
<thead>
<tr>
<th>Week</th>
<th>Max</th>
<th>Moe</th>
<th>Minnie</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>210</td>
<td>158</td>
<td>113</td>
</tr>
<tr>
<td>2</td>
<td>202</td>
<td>154</td>
<td>108</td>
</tr>
<tr>
<td>4</td>
<td>196</td>
<td>150</td>
<td>105</td>
</tr>
</tbody>
</table>

Make THREE different arguments….. each favoring a DIFFERENT dieter.
5.4 PROVIDE STUDENTS OPPORTUNITIES TO TEST THEIR THINKING

Documenting and testing student reasoning has to be built into the classroom culture. Teachers must begin to build this culture by allowing students to reason through problems without giving answers or validating ideas. When students understand that teachers trust pupils to come to appropriate conclusions by analyzing questions, talking with peers and checking for reasonableness they will engage in building their own schemas. Example Task 5.4 is one way in which this type of environment can be initiated. Che (2009) reports using this task on the opening day of school. One of the advantages of beginning with a reasoning task is that it requires students to rely on and examine their existing multiplicative schema. The idea that resolution to the questions will not be obtained for perhaps several weeks helps students begin to make the transition from computational thinkers to problem solvers. Students are forced into sharing their ideas and listening to and critiquing each other. Questions can be revisited as student schemas progress.

Example Task 5.4  Pencils (Che, 2009)

Posted around the room are four large pencils that a giant might use during mathematics class. If the giant uses pencils of this size, what can you find out about the giant?

Students should create a list of things they think they could find out and share them with each other and eventually a class list should be made.

Students could then work with partners to determine how they would answer one or some of the following questions:

<table>
<thead>
<tr>
<th>How tall is the giant?</th>
<th>How many times larger than you is the giant?</th>
</tr>
</thead>
<tbody>
<tr>
<td>How realistic is the pencil shape? Is it too wide compared with our pencils? Is it too long? Is the eraser an appropriate size?</td>
<td>How much does the giant weigh?</td>
</tr>
<tr>
<td>Could the giant walk in the door? Could he stand upright in the room?</td>
<td>How big would the giant’s desk need to be?</td>
</tr>
<tr>
<td>If you eat 2000 calories a day, how many calories might the giant need to eat?</td>
<td></td>
</tr>
</tbody>
</table>
Proportional tasks lend themselves to numerous solution methods and therefore provide good discussions to illustrate differing solution methods. Students need to be required to show their reasoning through illustrations and descriptions so they can present their solutions in small groups and whole class discussions. Labels are vital in working with ratios. Even when students use formal shortcut methods they must be able to articulate the reasoning behind their methods.

The research clearly indicates that student thinking is greatly influenced by the questions teachers ask. The need to ask questions about different perspectives has been documented by National Assessment of Educational Progress (NAEP). In the 1996 NAEP Population of Towns item, students were assessed to see if they could compare quantities additively (difference) as well as multiplicatively (by ratio). Town A and Town B grew by the same amount of people (3000) but by different percents (160% versus 150%). Only 1% of 8th graders and 3% of 12th graders correctly saw both claims. Only 20% of 8th graders and 25% of 12th graders provided partially correct answers that supported either the additive or multiplicative argument. According to Kilpatrick (2002) this result is because students have not had sufficient experiences for providing explanations and justifications for such arguments. Why? Because, teachers do not give them enough experiences.

6. PERSONAL REFLECTIONS

*Developing Essential Understandings* (Lobato, 2010) brings to light the important dialogue teachers need to engage in to deepen their understanding of the complexity of ideas associated with proportional reasoning. A teacher should build on a web of multiplicative ideas beginning in early multiplication and continuing past calculus. Understanding involves flexibility as a diversity of contexts involves differing schemas. Textbooks differ and often give rigidity to ideas where none exists. As teachers reason about ratios they are better able to
understand students’ misconceptions and to guide students through identified stages of reasoning (Langrall, 2000).

My personal experience has verified this important research finding that teacher reflection is critical. My autonomy in proportional problem solving made it difficult for me to describe either the process or the understandings that helped me solve problems. I was unaware of many of the essential components of underlying knowledge that allowed me to think proportionally. Clarity of thought about my own cognitive processes in solving proportional problems has been a result of my reasoning through ratio contexts. Studying the topics of relative thinking, unitizing, partitioning, interpretations of rational numbers, ratio sense making, quantities and change and other MCF topics has increased my ability to articulate the process and identify student “sticking” points.

As illustrated in Section 5 above, my synthesis of big ideas about developing proportionality will affect my teaching practices in both my presentation of tasks and my own questioning techniques. In addition I would like to highlight three other changes in my teaching practices that will result from my research: my emphasis on talking about change, my understanding of the use of definitions and my ability to more clearly articulate interconnections.

The identified pedagogical need to highlight change while emphasizing the constancy of a relationship is an important milestone in teaching proportional reasoning. Previous to this study I had never really considered how students knew about change. The research points out that students rely on their own interpretation of change to build natural schemas. When teachers merely tell students about change, their natural schemas remain unchanged. Students must be engaged in looking at their own schema and testing it with real world examples. This requires that an ongoing dialogue about change is present in the classroom.
Another important insight came in my quest to determine the “true definition” and distinctions between words such as ratio and rate or ratio and fraction. The research confirmed that the words were used inconsistently. The complexity of the subjects requires flexibility based on context. Giving simplified rigid definitions serves to restrict and hamper rather than help students master the concepts. The important ideas are articulated in the progressions for common core standards (2011) with flexibility: “A ratio associates two or more quantities.” “Ratios have associated rates.” “Equivalent ratios arise by multiplying each measurement in a ratio by the same positive number.” “Equivalent ratios have the same unit rate.” These definitions highlight the idea that ratios represent relationships. Proportional reasoning is not about formalized definitions but an understanding of the relationships that often come naturally to children. Capitalizing on children’s own experiences and giving them the ability to compare situations for these intuitive relationships is the key to developing proportional reasoning.

This change in emphasis from definitions to characterizing relationships has expanded my ability to interconnect proportional reasoning into the other topics of 7th grade. All computations imply some type of relationship. By studying the components of proportional reasoning, I have developed a flexibility and fluidity in which I can evaluate student responses. My increased sensitivity to students’ reasoning and desire for computational methods prior to quantitative understanding will help me scaffold upon their existing schemas and multiplicative perspectives. My own ability to separate conceptual operations from numerical operations should allow me to build schemas in which students can recognize and reason with reciprocal multiplicative relationships.
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