Understanding and Overcoming Students' Difficulties with Fractions: The Teacher's Role

David Kuralt

University of Utah
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**Introduction**

A fraction is an expression of the form \( \frac{m}{n} \), where \( m \) and \( n \) are integers, and \( n \neq 0 \). Thus a single fraction can be thought of as a quotient of integers. Why are fractions so difficult for students to learn, and so difficult for teachers to teach? In order to understand the difficulties that children have with fractions, it is necessary to know what they are asked to do with fractions.

The Common Core State Standards indicates that fractions are introduced to elementary students in the third grade. Working with visual models, and shapes in particular, students learn that if we divide a disc into four sectors that are the same size, and shade one of the sectors, then we have shaded one-fourth of the disc. Students learn to denote the quantity, one-fourth, with the symbol \( \frac{1}{4} \). A non-unit fraction, such as represented by \( \frac{3}{4} \), for example, represents three pieces of size one-fourth. See Figure 1.

![Figure 1](image1.png)

In addition to learning that fractions represent parts of whole quantities, third-grade students learn that proper fractions lie on the number line between 0 and 1. \( \frac{3}{4} \), for instance, can be located on the number line by partitioning the interval from 0 to 1 into four subintervals of equal size. The number \( \frac{3}{4} \) is located at the right end of the third subinterval to the right of 0. See Figure 2.
Third-grade students are also introduced to the idea of equivalent fractions. For example, we can use an area model to show children that the fractions $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent (Figure 3).

Students see that when the whole is partitioned into halves and one-half is taken, the amount taken is the same as when the whole is partitioned into fourths and two-fourths are taken. Third-graders also learn to compare two fractions with like numerators or like denominators. For example, students can work with visual models of fractions to show that $\frac{1}{3}$ is clearly less than $\frac{2}{3}$.

We also make use children’s intuition in getting them to see that if a whole is partitioned into fourths and another identical whole is partitioned into sixths, each fourth from one is larger than each sixth from the other. Thus $\frac{1}{6}$ is less than $\frac{1}{4}$. Finally, students learn that $\frac{n}{n} = 1$ and that $\frac{n}{n} = n$ for any natural number $n$.

In the fourth grade, students practice recognizing and generating equivalent fractions. Students extend their abilities to compare fractions. For example, in comparing the fractions


\[
\frac{2}{3} \text{ and } \frac{3}{4}, \text{ students are expected to generate the equivalent fractions } \frac{8}{12} \text{ and } \frac{9}{12}, \text{ respectively, in order to conclude that } \frac{2}{3} \text{ is less than } \frac{3}{4}. \text{ As another way of comparing fractions, students are encouraged to make use of what are called benchmark fractions. } \frac{1}{2} \text{ is an example of benchmark fraction. In comparing the fractions } \frac{2}{5} \text{ and } \frac{5}{8}, \text{ students are expected to recognize that } \frac{2}{5} \text{ is less than } \frac{1}{2} \text{ since } 2 \text{ is less than half of } 5. \text{ Similarly, students should recognize that } \frac{5}{8} \text{ is greater than } \frac{1}{2} \text{ since } 5 \text{ is greater than half of } 8. \text{ Thus students conclude that } \frac{2}{5} \text{ is less than } \frac{5}{8}.
\]

Fourth-graders learn to add and subtract fractions with like denominators. They are encouraged to view these operations as addition and subtraction of parts of the same whole. They also add and subtract mixed numbers with like denominators. Students learn that fractions can be decomposed into sums. For example, \( \frac{4}{5} = \frac{2}{5} + \frac{2}{5} \) or \( \frac{4}{5} = \frac{1}{5} + \frac{3}{5} \). Finally, fourth-graders evaluate the product of a unit fraction and a whole number, learning to view non-unit fractions as multiplicative quantities. For example, \( \frac{1}{4} \times 3 = \frac{3}{4} \) and \( \frac{2}{4} = 3 \times \frac{1}{4} \).

We are especially interested in the content of the fifth-grade curriculum, as set forth by the Common Core State Standards in mathematics, because in the last section of this paper, we will be designing a unit on fractions for fifth graders. Under the Common Core, it is in the fifth grade that students are introduced to the addition and subtraction of fractions with unlike denominators for the first time. Here, students again make use of their knowledge of equivalent fractions. In addition, students are also introduced to multiplication of two fractions. The algorithms for multiplying two fractions and for adding and subtracting fractions with unlike denominators are very different from each other. One of the challenges that many students face is keeping these algorithmic processes separated from one another.
Finally, students in the fifth grade are required to reason about fractions, developing what is called *number sense*. That is, students must develop the ability to determine whether or not the answers arising from their calculations with fractions are reasonable. A student with number sense, for example, would recognize that an answer of $\frac{2}{6}$ to the addition problem $\frac{1}{2} + \frac{1}{4}$ cannot possibly be correct. $\frac{2}{6}$ is clearly less than $\frac{1}{2}$, while the sum $\frac{1}{2} + \frac{1}{4}$ is clearly greater than $\frac{1}{2}$. A student with number sense also recognizes that multiplying any positive number by a positive quantity that is less than 1 results in a product that is less than the original number. Similarly, multiplying any positive number by a quantity that is greater than 1 results in a product that is greater than the original number. Students with number sense also know how to compare two fractions to determine if one is greater than the other, or if they are equivalent. They can show where these numbers lie on the number line in relation to the whole numbers and in relation to each other.

Beyond the fifth grade, and starting in the sixth grade, students are exposed to the idea of rational numbers as a *system* of numbers; that is, rational numbers comprise a *field*. Sixth-graders learn to divide fractions by fractions. The operation of division by a rational number is directly linked to multiplication by that rational number’s *multiplicative inverse*. Students learn the more formal ideas of *common multiples* and *least common multiple*, as would be applied to finding a common denominator for two or more fractions. Students also learn about *common factors*, and *greatest common factor*, as would be applied to identifying equivalent fractions, and in reducing fractions to lowest terms. In addition, the role of rational numbers in describing ratios and proportional relationships sees greater emphasis, and students begin to build a conceptual foundation for many ideas in algebra and geometry. In theory at least.
It is well documented that students in middle school and high school are lacking many of the skills with fractions that they ideally would have acquired in elementary school (Petit, Laird, & Marsden, 2010). Why is this? What is so hard about fractions? Why do some students persist in adding numerators and denominators directly to conclude that \( \frac{1}{2} + \frac{1}{4} = \frac{2}{6} \)? Why do other students waste time trying to find a common denominator when multiplying fractions? Why is it that still other students cannot determine the two consecutive whole numbers between which \( \frac{12}{7} \) lies? As we shall see, the answer to these questions may be that fractions are inherently more complicated than whole numbers. By “more complicated,” we mean that the concept of fraction encompasses many possible interpretations (Kieren, 1976) and representation modes (Lesh, Post, & Behr, 1987). By contrast, whole numbers are easily visualized as everyday objects, such as apples, and the arithmetic operations on whole numbers can be conceptualized in terms of apples, for example. We can add \( m \) apples to \( n \) apples. We can take away \( m \) apples from \( n \) apples. We can determine how many apples there are if we have \( n \) bags with \( m \) apples in each bag. We can determine how many bags it will take to bag \( n \) apples when we know that each bag will hold \( m \) apples. However, in order to perform these same arithmetic operations on fractions and to understand what they are doing in the process, students need to make decisions about how to interpret and represent the fractions. What does it mean to add \( \frac{3}{4} \) to \( \frac{1}{2} \), or to multiply the fractions \( \frac{3}{7} \) and \( \frac{2}{5} \)? Is it most helpful to interpret the fraction \( \frac{2}{5} \) as the quotient \( 2 \div 5 \), or should we regard \( \frac{2}{5} \) as a spot on the number line that lies a little to the left of \( \frac{1}{2} \)? Is it more helpful to draw pictures to evaluate the product \( \frac{3}{7} \times \frac{2}{5} \), or does it make more sense to construct a story in which one is called upon to find the area of a rectangle?
Complicating the proposition of working with fractions even further is that each fraction can be expressed by any of infinitely many equivalent fractions. When early elementary students were working exclusively with whole numbers, they were never required to choose a symbolic representative of any number from a multitude of choices. When students worked with the number three, for instance, the only symbol ever needed was 3. Now, when working with the quantity one-half, students must sometimes choose the most appropriate symbolic representative. Do we use $\frac{1}{2}$, $\frac{3}{6}$, or $\frac{5}{10}$? Students working with fractions must also understand that each whole number has infinitely many possible symbolic representations as fractions. They must sometimes choose among the symbols $3, \frac{3}{1}, \frac{6}{2}$, and more.

We now review what the literature has to say about teaching fractions in elementary and middle school. We discuss the different representation modes of fractions and what working within some of these modes teaches us about children’s thinking. We identify the difficulties of learners not only as they struggle to work problems with fractions, but as they struggle to understand the notion of fraction, itself. We describe some of the problems that students have with number sense and fraction arithmetic. We also discuss some ideas for effective instruction that are backed by empirical research.

**Literature Review**

**How Pictorial Representations Reveal Children’s Misconceptions**

Richard Lesh, Tom Post, and Merlyn Behr describe five representation modes for fractions (Hodges, Cady, & Collins, 2008). They are “1) real-world contexts, 2) pictures, 3) written language, 4) manipulatives, and 5) symbols” (Hodges et al., 2008, p. 79). We will discuss pictorial representations in this section, and we will discuss real-world and symbolic representations in the following section. We will defer the discussion of the written language and
When children are first introduced to fractions in the third grade, most of them work with visual models of fractions. One such model is called the area model. Figure 1 is an example of an area model. Students are typically asked what portion of the disc is shaded. For the disc on the left, the correct response is “one-fourth,” and for the disc on the right, it is “three-fourths.” So one of the first things that students are taught is that fractions describe part-to-whole relationships. The four equal-sized quadrants of the disc comprise the whole, and the shaded quadrants constitute the part. Students are also taught the conventional symbolic representation for fractions. Thus, one-fourth is written $\frac{1}{4}$ and three-fourths is written $\frac{3}{4}$.

Another visual model that illustrates part-to-whole relationships is the set model. In this model, a set of several identical objects makes up the whole, and we are interested in a particular subset of the objects. For example, suppose we have four rubber balls that are identical in every way except for color. Three are green and one is red. See Figure 4.

The set of four balls is the whole, and we say the three-fourths of the balls are green and one-fourth is red. Another visual mode that Hodges describes is the length model. This model is useful in comparing the lengths of two objects, or in measuring the length of an object against a
standard length. For example, one might use a ruler to measure the length of a ladybug and conclude that it is one-fourth of an inch long. See Figure 5.

As intuitive as such models may seem, we get a glimpse of the difficulties that can arise when we make a small change to one of these models. Consider what can happen when we look at an area model in which the parts are not of equal size. As an example, we again use a disc. See the figure below. Again we ask a child to tell us what portion of the disc is shaded. Geoffrey B. Saxe observes that some students tend to disregard the size of the parts and assume that each part contributes equally to the whole (Saxe, Taylor, McIntosh, & Gearhart, 2005). Thus a student answering our question about what portion of the disc is shaded might respond “one-third.”

Another common visual model is the number line, which is a special form of the length model. In this model, we not only identify all of the whole numbers, but we can identify all of
the fractions that lie between the whole numbers as well. Meghan Shaughnessy identifies a number of difficulties that children encounter when asked to identify fractions on a number line (Shaughnessy, 2011). In her study of students in upper elementary grades, she asks them to identify fractions that lie on a number line. Shaughnessy’s study reveals four basic categories of error. The first involves the use of unconventional fraction notation, which is the fifth of our five representation modes of fractions. We will say more about conventional notation later. The second common error type is what Shaughnessy calls *redefining the unit*. Shaughnessy describes an example in which the interval between 0 and 2 is partitioned into twenty subintervals of equal length. The student labeled the place for $\frac{8}{10}$ as $\frac{8}{20}$, having counted twenty subintervals between 0 and 2, and having identified the indicated tick mark as being the end of the eighth subinterval. Here, the student regarded the whole, or unit, not as the interval between 0 and 1, but as the entire interval from 0 to 2. See Figure 7.

![Figure 7](image-url)

The next basic error type is what Shaughnessy calls a *two-count strategy*. As an example of this type of error, consider the interval between 0 and 1 divided into five equal subintervals. A student making a two-count error counts the discrete number of tick marks in the *entire* interval to use in the denominator, and then counts the number of discrete tick marks leading to the indicated place. Thus $\frac{2}{5}$ would be labeled as $\frac{3}{6}$. See Figure 8.
The fourth and final error type Shaughnessy discovered is what she calls a *one-count strategy*. In this type of error, students “count discrete quantities in ways that do not attempt to coordinate the number of discrete quantities from zero to the target point with the number of discrete quantities from zero to one” (Shaughnessy, 2011, p. 433). She gives an example of this strategy in which a student counted the number of intervals from 0 to the indicated place and put this number in the denominator. For the numerator, he simply chose the number 1 since the indicated place was nearest to the whole number 1. Below is an example of how an error of this type could look.

![Figure 8](image)

In her article, Shaughnessy cautions that number line tasks that are not designed carefully can conceal misconceptions that students have about fractions on a number line. She goes on to
give pointers for designing tasks that reveal students’ thinking, thereby allowing teachers to address misconceptions.

**How Symbolic Representations and Real-World Contexts Reveal Children’s Misconceptions**

Students learning about fractions for the first time often struggle to establish a link between their intuitive understanding of the idea of “half,” for instance, and the conventional way of writing one-half, which is $\frac{1}{2}$. Conversely, children may not know to what the symbol $\frac{1}{2}$ refers. Earlier we discussed an example of a one-count strategy error from Megan Shaughnessy’s work with children. The student counted the number of tick marks to the indicated place, and put this number in the denominator, while choosing the nearest whole number, 1, for the numerator. Geoffrey Saxe would describe this error as a problem with reference (Saxe et al., 2005). To what quantity does the numerator refer? To what quantity does the denominator refer? The answers to these questions are further complicated by the possibility that when confronted by an area model, as opposed to a number line, the same student who uses a one-count strategy on the number line might correctly identify the part and the whole in the area model, and might correctly represent this part-to-whole relationship in conventional fraction notation. In other words, the quantities to which the numerator and denominator refer when a child writes a fraction may be as much a function of the visual model used as it is a function of the child’s grasp of part-to-whole relationships.

Children’s struggles with symbolic representations of fractions are also evident in real-world contexts. In her work with seven children in the third and fourth grades, Nancy Mack assessed the ways in which these students’ informal knowledge shaped the ways in which they interpret symbolic fractions (Mack, 1995). As Mack describes it, *informal knowledge* “can be
characterized generally as applied, real-life circumstantial knowledge constructed by the individual student. This knowledge may be either correct or incorrect and can be drawn on by the student in response to problems posed in the context of real-world situations familiar to him or her” (Mack, 1995, p. 422-3). None of these students had received much formal instruction in fractions. Mack posed problems to the children that were based in simple real-world contexts, and asked them to write fraction responses to her questions. Mack found that student errors were one of two basic types of error. Students either “explained symbolically represented fractions in terms of whole number quantities,” or “explained symbolic representations for whole numbers in terms of fractional quantities” (Mack, 1995, p. 429). As an example of the former situation, Mack asks Amy, one of her students, the following question: “…if there’s a whole pizza and I say you can have one-third of the pizza, how much do you get?” (Mack, 1995, p. 431). Amy correctly responds, “one of three pieces.” Mack follows up by pointing to the symbol \(\frac{1}{3}\), written earlier on Amy’s paper, and asks, “So what does that mean? That one-third?” Amy responds, “Three pieces or one pizza that’s cut into three pieces” (Mack, 1995, p. 431). While Amy was able to draw on her informal knowledge to answer the verbal question correctly, she ran into difficulty when asked to interpret the symbol \(\frac{1}{3}\). In Amy’s mind, the numerator of \(\frac{1}{3}\) referred to one whole pizza, while the denominator referred to the number of pieces into which the pizza was cut.

Mack encountered the second type of error when she moved on to problems involving subtracting a fraction from a whole number. Ted was able to answer correctly when asked how much pizza would be left if we subtract four-fifths of a pizza from a whole pizza. However, when presented with the subtraction problem \(1 - \frac{4}{5}\), he responded initially that the answer was \(\frac{3}{5}\), “…’cause you took one piece of my pizza, then I had four—I mean five pieces and I had four
and you took one away, so I had three left and there’s still five out of five pieces” (Mack, 1995, p. 434). Ted did not regard the 1 in the problem $1 - \frac{4}{5}$ as one whole pizza, but rather regarded the 1 as one of the five slices of pizza. As a result, he simply subtracted 1 from 4 to get 3, or $\frac{3}{5}$ of a pizza.

**Children’s Struggles with Fraction Arithmetic and Number Sense**

Many students at the elementary, middle school, and high school levels exhibit severe difficulties in adding, subtracting, multiplying, and dividing fractions and mixed numbers. A *mixed number* is a numerical expression in which a whole number component appears side-by-side with a fractional component. The expression $1 \frac{2}{3}$ is an example of a mixed number. Researchers have attributed some of the difficulties that students have to their tendency to apply their knowledge of the arithmetic of whole numbers algorithmically to fractions (Mack, 1995; Howard, 1991). For example, when presented with an addition problem of two fractions, many children simply add the numerators to obtain the numerator of the sum, while adding the denominators to obtain the denominator of the sum. Thus many children conclude that $\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$.

This practice is further reinforced by the algorithm for multiplying two fractions, where one actually does multiply the numerators and multiply the denominators to obtain the numerator and denominator, respectively, of the product.

Another contributing factor to students’ difficulties with fractions is the traditional emphasis that is placed on algorithms for the performance of whole-number arithmetic. We learn to compute sums of two whole numbers by positioning one above the other so that the one’s digit, tens digit, hundreds digit, etc. of each whole number all align vertically. We “carry the 1” to the next column on the left when the sum of the digits in our immediate column exceeds 10. The algorithms for subtraction, multiplication, and division of two whole numbers are just as
prescriptive. Students who perform these algorithms fluently and correctly are considered by politicians, school administrators, teachers, and parents alike, to have “mastered” arithmetic. It stands to reason that a child who performs these arithmetic algorithms fluently and correctly would consider himself to have mastered arithmetic as well. Thus adding the numerators and denominators of fractions seems perfectly reasonable for these students (Howard, 1991).

The emphasis on algorithm carries over to fraction arithmetic, where students are taught that when we add two fractions with like denominators, we add the numerators to determine the numerator of the sum, but we leave the denominator alone. That is, the denominator of the sum is the same as the denominator of the addends. Symbolically, this algorithm seems odd to students who are accustomed to adding columns of whole numbers. We show students why leaving the denominator the same is reasonable by using a pictorial representation of what is happening, such as by drawing a pizza cut into six equal-sized slices, and showing students that if José eats two slices ($\frac{2}{6}$ of the pizza) and Maria eats one slice ($\frac{1}{6}$ of the pizza), then together they have eaten three slices, or $\frac{3}{6}$ of the pizza. The problem is that students are often presented problems like $\frac{2}{6} + \frac{1}{6}$ without an accompanying real-world context in which to understand what is going on (Hodges et al., 2008; Charalambous, Delaney, Hsu, & Mesa, 2010). Thus students, when presented with an addition problem of two fractions, will simply add the numerators and denominators even though they know this procedure is wrong (Howard, 1991; Vinner, Hershkowitz, & Bruckheimer, 1981). Students also commonly apply the wrong algorithm to a fraction arithmetic problem. For example, many students will attempt to find a common denominator when multiplying two fractions. Ironically, it is the algorithm for multiplication that calls simply for the multiplication of numerators and the multiplication of denominators.
Students misconceptions of symbolic reference, combined with their tendency to misapply algorithms to fraction arithmetic, leads to additional difficulties in number sense. As we mentioned in our introduction, number sense is the ability to determine whether or not our numerical answers to problems are reasonable. A student who uses the wrong algorithm to conclude that $\frac{1}{2} + \frac{1}{4} = \frac{2}{6}$ typically does not stop to consider whether this answer is reasonable or not. It does not occur to him that $\frac{2}{6}$ cannot possibly be the right answer because $\frac{2}{6}$ is less than $\frac{1}{2}$. Students who lack number sense have trouble determining where two fractions would lie on the number line relative to each other and relative to the whole numbers. Such a student might conclude that $\frac{1}{3}$ is greater than $\frac{1}{2}$ because 3 is greater than 2. Here, the student relies solely on symbols to reach his conclusion, and has no understanding of quantities that the symbols represent.

By the time children reach the upper elementary grades, they have already seen a number of contexts in which fractions apply. Some of these contexts are concrete, and appeal to children’s informal knowledge, such as when considering an equal sharing problem. Other contexts are more abstract, such as when children try to represent an equal sharing problem symbolically, or try to compare two symbolic fractions to determine which is greater. We also want children to understand that a fraction represents a single quantity, and that this quantity can be thought of as lying somewhere on the number line. We want children to understand what the numerator and denominator of a fraction mean in any context. We would even like children to understand that across contexts, fractions “behave” the same way. For example, children know that if a pizza is cut into eight pieces of equal size, and three of those pieces are eaten, we are left with less than one whole pizza. Likewise, we hope that children will recognize that the symbolic problem $1 - \frac{3}{8}$ has an answer that is less than 1. Finally, we want children to build schemas in
which fractions and whole numbers are understood to be part of the same number system, and
that they interact with each other in meaningful ways. For example, the same notion of order that
children seem to understand intuitively for whole numbers applies to fractions as well. Where we
have $2 < 3$ and $\frac{1}{3} < \frac{1}{2}$, we also have $1 < \frac{5}{4}$.

**Ideas on How to Teach Fractions**

Thomas Kieren recommends incorporating the various interpretations of rational numbers
in instruction (Kieren, 1976). Before we begin, it is appropriate to define and discuss these
interpretations in some detail. An *interpretation* is a schema in which rational numbers can be
understood. Kieren describes seven interpretations of rational numbers while acknowledging that
his list is not exhaustive, and the interpretations are not mutually exclusive.

As we described in our introduction, a *fraction* is an expression of the form $\frac{a}{b}$, where $a$
and $b$ are integers and $b \neq 0$. A *rational number* is any number that can be expressed as a
fraction. Thus a fraction is one way of representing a rational number symbolically. Kieren first
describes *rational numbers as fractions*, in which fractions are viewed as “objects of calculation”
(Kieren, 1976, p. 104). In this interpretation, one learns to work with fractions by remembering
specific definitions and algorithms. Fraction addition problems, for example, are categorized. For
each category, a procedure has been devised. We know how to add two fractions with like
denominators. We know how to add two fractions with unlike denominators. We know how to
add two mixed numbers or two improper fractions. While fluency in performing operations on
fractions depends on a strong command of the relevant algorithms, children are unlikely to
remember how to do these operations if the curriculum emphasizes algorithmic proficiency over
understanding. Many children will forget what they have learned as soon as their class moves on
to the next mathematical topic.
Kieren then describes *rational numbers as equivalence classes of fractions*. Here, two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. An individual rational number, then, can be viewed as a set of all equivalent fractions. The number, one-half, can be represented as $\left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots \right\}$. Another closely related interpretation is that of *rational numbers as ratio numbers*. In this interpretation, rational numbers are expressed as ordered pairs. Thus, one-half is expressed as (1,2). The notion of equivalency applies here as well, where for example, $(1,2) = (3,6)$ because $1 \times 6 = 2 \times 3$. Understanding the notion of equivalent fractions is key to understanding the algorithm for adding two fractions with unlike denominators.

Another interpretation of rational numbers is *rational numbers as operators or mappings*. Within this interpretation lies the framework for understanding what it means to multiply by a fraction. Imagine a line segment. Applying the operator $\frac{1}{2}$, to this line segment results in a line segment that is half the length of the original. If we apply the operator, $\frac{5}{4}$, to the line segment, we obtain a line segment that is $1 \frac{1}{4}$ times the length of the original. We can also imagine the numbers on a number line as line segments. 6, for instance, can be seen as the line segment whose endpoints are 0 and 6 on the number line. Multiplying 6 by $\frac{1}{2}$ yields 3. Multiplying 6 by $\frac{4}{3}$ yields 8. The product $\frac{1}{2} \times \frac{1}{4}$ can be viewed as a composition of operators. We apply $\frac{1}{2}$ to the line segment whose endpoints are 0 and 1, and apply $\frac{1}{4}$ to the resulting line segment. The end result is the line segment whose endpoints are 0 and $\frac{1}{8}$. Thus $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

Another interpretation is that of *rational numbers as elements of a quotient field*. That is, given any three rational numbers $a$, $b$, and $c$, and two binary operators $+$ and $\times$, called *addition* and *multiplication*, respectively, we have:
1. $a + b$ and $a \times b$ are also rational numbers (closure of addition and multiplication)

2. $a + b = b + a$ and $a \times b = b \times a$ (commutativity of addition and multiplication)

3. $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$ (associativity of addition and multiplication)

4. $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$ (distributivity of multiplication over addition)

5. There are two rational numbers, denoted $0$ and $1$, called the additive identity and multiplicative identity, respectively, such that for any rational number $a$, we have $a + 0 = 0 + a = a$ and $a \times 1 = 1 \times a = a$

6. Associated with each non-zero rational number $a$, we have two rational numbers, denoted $-a$ and $\frac{1}{a}$, called the additive inverse and multiplicative inverse, respectively, such that $a + (-a) = (-a) + a = 0$ and $a \times \left(\frac{1}{a}\right) = \left(\frac{1}{a}\right) \times a = 1$. Note that $0$ is its own additive inverse since $0 + 0 = 0$, and that $0$ has no multiplicative inverse since the product of $0$ and any other rational number is always $0$.

While we do not expect elementary and middle school children to memorize and identify the field axioms, we do hope that by the time children enter high school, their intuition of fractions tells them that

1. The sum or product of any two fractions exists.

2. The sum or product of two fractions is unique regardless of the order in which they are written. For example, $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. 
3. We can add or multiply more than two fractions in any order we like. For example,
\[ \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}, \quad \text{and} \quad \frac{1}{4} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{24}. \]
This last example illustrates a combination of the commutative and associative properties of multiplication.

4. When we add 0 to any number, or multiply any number by 1, the number remains unchanged.

5. Subtraction of any number from itself yields 0. All fractions, except for 0, have reciprocals, and the product of any fraction and its reciprocal is 1.

6. Integers are fractions, and the algorithms we use to add and multiply fractions are consistent with the algorithms we use to add and multiply integers. For example,
\[ \frac{2}{5} + \frac{1}{5} = \frac{3}{5}, \quad \text{while} \quad 6 + 3 = \frac{6}{1} + \frac{3}{1} = \frac{9}{1} = 9, \quad \text{and} \quad \frac{2}{5} \times \frac{1}{5} = \frac{2}{25}, \quad \text{while} \quad 6 \times 3 = \frac{6}{1} \times \frac{3}{1} = \frac{18}{1} = 18. \]

Still another interpretation of rational numbers is rational numbers as measures. As we did in explaining the way in which fractions can be viewed as operators, we associate with each rational number a line segment, which lies on the number line, and whose endpoints correspond to 0 and the rational number, itself. We can then visualize the addition of two positive fractions as the concatenation and translation of the line segments so that one endpoint of the augmented segment corresponds to 0, while the other corresponds to the sum of the two fractions. See Figure 10.
We will not discuss the interpretation of *rational numbers and decimal fractions* in great depth here, although it is appropriate to mention the role of decimal numbers in the study of rational numbers. Under the Core Curriculum, children in the fifth grade will also be working with decimals numbers. Crucial to children’s understanding of rational numbers will be their ability to recognize the connections between the decimal and fraction representations of rational numbers, and the ability to convert a rational number from one representation to the other. Children will also need to understand that the bases of the algorithms for performing arithmetic on fractions and on decimals are linked.

Within the different interpretations of rational numbers that Kieren discusses lie some crucial ideas that anyone must assimilate in order to understand fractions. We list some of these ideas here:

1. A given fraction has an infinite number of equivalent symbolic representatives.
2. Rational numbers, as represented by fractions, comprise a *field*. That is, there is a set of rules that fractions, as a whole, obey. These rules are embodied by the operations of addition and multiplication and the properties as articulated by the field axioms we discussed earlier.
3. Algorithms exist for adding and multiplying fractions.
4. Multiplication by a fraction serves as an operator to rescale a quantity, either making the quantity greater than or less than it was originally.
5. Addition of fractions can be envisioned as the concatenation of line segments.

In addition to working and becoming familiar with the various interpretations of fractions, it is important to engage students at different levels of cognitive demand in the tasks they are given. Mary Kay Stein proposes four levels of cognitive demand (Stein, Smith,
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Henningsen, & Silver, 2009). The least demanding tasks fall into the category of *memorization tasks*. These tasks include the recollection and recitation of previously learned facts. There are no algorithms to follow. Students either know the correct answer, or they do not. An example of such a task would be a true-or-false prompt that states “We need to find a common denominator in order to multiply fractions.” All that is required of the students is the answer “false.”

A *procedures without connections* task involves arriving at the correct answer by properly executing an algorithm. An example of such a task is the addition problem $\frac{2}{5} + \frac{1}{3}$. The procedure is to convert each of the fractions in the sum into equivalent fractions with like denominators, and rewrite the problem as $\frac{6}{15} + \frac{5}{15}$. Then we carry out the addition and write the answer $\frac{11}{15}$. No explanation is requested, and that answer is either right or wrong.

A *procedures with connections* task requires more decision-making on the part of the student. An algorithm is involved in solving the problem, but the student must determine which algorithm to apply, and to which objects the algorithm must apply. Word problems found in textbooks are common examples of such tasks. *Jill owns two Kentucky Fried Chicken franchises and Carlos owns three. How many times the number of Jill’s franchises does Carlos own?* The numbers 2 and 3 must interact, but how? It is left to the students to determine that the answer must be the ratio of the number of franchises that Carlos owns to the number of franchises that Jill owns. Furthermore, students must reason, whether the problems asks them to or not, that $\frac{3}{2}$ is a reasonable answer because $\frac{3}{2}$ times the number of franchises that Jill owns, 2, is equal to 3. In symbols, $\frac{3}{2} \times 2 = 3$.

Stein refers to the tasks of the highest cognitive demand as *doing mathematics tasks*. Such tasks “require complex and non-algorithmic thinking” (Stein et al., 2009, p. 6). Students
must explore their own knowledge of mathematical ideas and the connections that exist between these ideas in order to solve the problem. An example of a doing mathematics prompt would be *Identify a two-dimensional geometric shape that is good for modeling fractions. Explain why this shape is effective.*

New teachers, especially, rely heavily on textbooks in designing instructional activities in fractions. Researchers have found, however, that exercises and tasks in many textbooks do not require students to function at the doing mathematics level, or even at the procedures with connections level (Hodges et al., 2008; Charalambous et al., 2010). In a study of three American textbooks, Thomas Hodges compares the emphasis placed on each of the five representation modes of fractions. By far, most of the exercises made use of the symbolic mode only, and “most of these problems were also void of context” (Hodges et al., 2008, p. 80). The issue of low cognitive demand is not unique to American textbooks, either. Charalambous and his colleagues compare the treatment of addition and subtraction in textbooks from Cyprus, Ireland, and Taiwan, all small island nations with federally mandated curricula. The textbooks in their analysis were designed for children at levels equivalent to the American fourth or fifth grade. The study found that tasks in the Irish and Cypriot textbooks made relatively low cognitive demand on the students. That is, the tasks usually called for a numerical response that could be arrived at by applying the correct algorithm for addition and subtraction of fractions. There were few, if any, tasks that required students to explain their reasoning. By contrast, the Taiwanese texts called for a variety of responses not merely limited to answers of fraction addition and subtraction problems. For example, “Even tasks that required procedures without connections in the Taiwanese textbooks expected students to find missing addends…” (Charalambous et al.,
Taiwanese textbooks, then, seem to be the exception to the rule of textbooks dominated by procedures without connections.

In summary, we must design teaching activities for children in which they must work within different interpretations and within different representations modes of fractions. We must require students to move from one interpretation or representation mode to another. In addition, we must expose children to tasks that place varying levels of cognitive demand on them. While we want children to compute fluently with fractions, we also want them to be able to explain to their teachers and to each other why the computations work. We want students to gain experience in looking for connections between seemingly unrelated ideas. For instance, in addition to hoping that children will be able to calculate the sum $\frac{5}{16} + \frac{5}{12}$, we would also like children to understand and be able to explain how to find common multiples of 12 and 16 in order to find a common denominator. We want children to understand and be able to explain why a common denominator is desirable. Finally, we want children to understand and be able to explain that we multiply each of the fractions by an appropriate form of 1 to obtain equivalent fractions that have the desired common denominator. In the next section of this paper, we design a teaching unit on fractions for fifth graders. We will implement the specific recommendations of this section to design instruction that will help students cultivate the understandings we have described here.

**A Teaching Unit on Fractions for Fifth Graders**

This section of the paper focuses on the construction of a teaching unit on fractions for children in the fifth grade. This unit is based the content standards of the Common Core. The design of this unit is based on the principles of *backward design*, in which we specify the desired learning outcomes of the unit, as well as how we intend to assess student learning, before we so
much as write the first lesson plan (Wiggins & McTighe, 2005). Our guiding principles in this unit will be working within and between different interpretations and representation modes of fractions. We will place as much emphasis on requiring children to explain their thinking as on their procedural fluency.

As we stated earlier, we are interested in the curriculum for the fifth grade under the Common Core because the fifth grade is the first time that children are required to add and subtract fractions with unlike denominators, and children are required to multiply fractions by fractions. The algorithms for adding fractions and multiplying fractions are very different from one another. If addition and multiplication of fractions are not taught carefully, and are not taught for understanding, then students are doomed to misapply the algorithms or forget them altogether (Vinner et al., 1981). But what does it mean to understand?

**Six Facets of Understanding**

Grant Wiggins and Jay McTighe propose six facets of understanding (Wiggins & McTighe, 2005). We shall construct a definition of understanding based on these six facets. The first facet is explanation. A child exhibiting this facet might explain that the fraction $\frac{9}{12}$ represents the same quantity as the fraction $\frac{3}{4}$ by drawing a figure such as Figure 11.
Teachers can help children to cultivate this facet of understanding by designing learning activities in which students teach each other.

The second facet of understanding is *interpretation*. Children with this type of understanding have an appreciation for the subject matter because it is immediately relevant to them. A child who realizes that three-fourths of an inch corresponds to the same location on the ruler as twelve-sixteenths of an inch has a deeper appreciation of the fact that \( \frac{3}{4} \) and \( \frac{12}{16} \) are equivalent fractions.

The third facet of understanding is *application*. “Understanding involves matching our ideas, knowledge, and actions to context” (Wiggins & McTighe, 2005, p. 93). In other words, application is the ability to use what we know in a real-world context. A student working on a project in automobile shop class tries to loosen a bolt with a socket of size \( \frac{11}{16} \) inches. The socket does not fit, however, because it is slightly too small. The student then looks for a \( \frac{3}{4} \)-inch socket because he realizes that \( \frac{3}{4} = \frac{12}{16} \), and his knowledge of sockets tells him that socket measurements are given in the lowest terms.

The fourth facet of understanding is *perspective*. Perspective is the ability to see an issue from multiple points of view. A student with perspective is able to look at an issue dispassionately. She can set aside, for the moment, her own feelings and opinions. Such a student does not necessarily view the symbolic representation \( \frac{3}{4} \) as being superior to \( \frac{12}{16} \), even though textbooks and teachers alike have placed, at times, an obsessive emphasis on expression fractions in the lowest whole-number terms. The student knows that there are situations in which the representation \( \frac{12}{16} \) might even be preferable to \( \frac{3}{4} \), such as when adding fractions whose common denominator is 16.
The fifth facet of understanding is *empathy*. “Empathy is the deliberate act of trying to find what is plausible, sensible, or meaningful in the ideas and actions of others, even if those ideas and actions are puzzling or off-putting” (Wiggins & McTighe, 2005, pp. 98-99). In other words, empathy is the ability to see the point of view of another person. While it might be obvious to one student that \( \frac{3}{4} = \frac{12}{16} \), this fact might not be obvious to her classroom neighbor. In the conversation that takes place with her neighbor, the student gains insights into the sticking points that her neighbor is experiencing in understanding equivalent fractions. She gains empathic understanding of her neighbor’s point of view, and a deeper appreciation of her own understanding of equivalent fractions.

The sixth facet of understanding is *self-knowledge*. A child’s self-knowledge is his awareness of how he learns best, what motivates him, what confuses him, and what strategies of study help him most when he is confused about something. We gain self-knowledge experientially over time. A student who did not like fractions last year finds this year that he is catching on more easily. In reflecting on this difference, he realizes that his command of multiplication facts last years was not as good as it is this year. So this year, he finds it much easier to generate equivalent fractions.

Wiggins and McTighe argue that a child with facilities in multiple facets of understanding understands more deeply than a child who, for instance, can work a fraction addition problem but cannot explain the steps of the procedure. As a means of setting forth a definition of understanding, we say that children who understand:

1. Can explain why an idea is true.
2. Appreciates the idea’s relevance to their own lives.
3. Can apply the idea in different problem-solving contexts.
4. See the idea from multiple points of view.

5. Appreciate different approaches to the same problem, and how some students may perceive difficulty.

6. Can analyze their own reactions to, and struggles with, the idea.

The successful design of our unit on fractions for fifth-graders depends upon knowing clearly what understandings we want students to have as a result of working through our unit. Identifying these understandings is the first step in a process called backward design, as set forth by Wiggins and McTighe. We shall use the principles of backward design to build our teaching unit on fractions for fifth-graders.

According to Wiggins and McTighe, the “twin sins” of traditional design are building a teaching unit around activities, or around coverage of a specified set of topics. Activities by themselves may be fun and engaging, but do not necessarily lead to shared understandings among students. Teaching for coverage of certain mathematical topics can limit chances students have to explore connections to other topics in the unit or even to topics across units or subjects. Furthermore, coverage-based instruction tends to leave in the dust students who do not immediately grasp the topics being covered, as topics tend to build sequentially one upon the prior one.

Backward design is backward in the sense that it first focuses on desired learning outcomes. Before any meaningful activities or assessments are designed, teachers need to identify explicitly what it is that students must learn. What are the desired learning outcomes of this unit? Next, teachers must figure out how to determine whether or not these desired learning outcomes are being accomplished. It is only after the goals of the unit and the means of
measuring accomplishment of these goals are determined that the teacher can begin designing lessons and activities for students. We shall now work through these three steps:

1. Identify desired learning outcomes.
2. Determine acceptable evidence of learning.
3. Design learning activities and lessons.

**Stage 1: Identifying Desired Learning Outcomes.**

Wiggins and McTighe advocate identifying and classifying unit goals in three broad categories: *Big ideas and core tasks*, *Important to know and do*, and *Worth being familiar with*. To help us articulate some of these goals, we will make use of the content standards of the Common Core State Standards Initiative (www.corestandards.org).

**Big Ideas:**

- \( \frac{n}{n} = 1 \) for any natural number \( n \).
- There are infinitely many equivalent symbolic representations for any given fraction.
- Multiplying any quantity \( q \) by a fraction greater than 1 produces a quantity greater than \( q \). Alternatively, multiplication by a fraction less than 1 produces a quantity less than \( q \) (Standard 5.NF.5). A related idea is that multiplication by 1 produces no change to a quantity, and this is the central idea of generating equivalent fractions.
- The fraction \( \frac{a}{b} \) can be interpreted as the quotient \( a \div b \) (Standard 5.NF.3).
- \( a \div b = c \) and \( c \times b = a \) are equivalent statements. We use this fact to explain division of a unit fraction by a whole number and division of a whole number by a unit fraction (Standard 5.NF.7).
Core Tasks:

- Generate equivalent fractions in order to add and subtraction fractions with unlike denominators (Standard 5.NF.1), and also to compare two fractions with unlike denominators.
- Reason about the size of a product as compared to the size of the factors without carrying out the multiplication. For example, $3 \times \frac{5}{4}$ is greater than 3 because $\frac{5}{4}$ is greater than 1.
- Interpret and solve word problems, choosing the correct symbolic representations for any fractions, and choosing the necessary operations.

Important to Know and Do:

- Determine what the common denominator should be when adding, subtracting, and comparing fractions with unlike denominators. Students must know how to find a common multiple of two natural numbers. The product of the denominators is a natural choice for the common multiple.
- When multiplying fractions, we multiply the numerators to obtain the numerator of the product, and we multiply the denominators to obtain the denominator of the product.

Worth Being Familiar with:

- Multiplication facts through 12.
- How to find a least common multiple of two natural numbers.
- Convert a mixed number to an improper fraction and vice versa.
- How to reduce a fraction to lowest terms. Rules for divisibility by natural numbers 1-10.

Stage 2: Determining Acceptable Evidence of Learning.

The main purpose of assessment is to measure the extent to which students are achieving our desired learning outcomes. Therefore, assessments are directly linked to the goals that we set.
forth in Stage 1. Not all assessments are graded, such as the brief formative assessments that teachers conduct during a class lesson to check for student understanding and engagement.

When the assessment is graded, it is helpful to state explicitly what is being measured. To this end, we employ a *rubric*. A rubric is a list of achievement standards, and the student’s response is graded according to each standard. Here is an example of an assessment prompt and a rubric to grade it. The problem in the example below is worth two points, and the rubric spells out specifically how those two points are to be earned.

*You are working with a set of wrenches whose sizes are given in fractions of an inch. Explain why you do not find a wrench labeled “4/16.” If there is a wrench of this size, how would it be labeled?*

**Explanation contains the idea of equivalent fractions (1 point):** 0 points if explanation makes no sense or does not contain the idea that fractions can be represented symbolically in more than one way. 1 point if explanation contains the idea of equivalent fractions.

**Generation of equivalent fractions (1 point):** 0 points if student provides no answer to the second part or if the response makes no sense. 1 point if the students generates a fraction that is equivalent to $\frac{1}{4}$.

In addition to measuring the extent to which students are achieving our learning goals, we can also use assessment to inform different aspects of our instruction. Such assessments can help us to answer questions such as:

- Is the pace of instruction overwhelming my students?
- Are the homework exercises sufficiently challenging?
- Do we need to revisit certain ideas that we have discussed previously?
The following is an example of an assessment prompt that can help us answer the questions above:

*What was the most difficult mathematical task you encountered in the last couple of days?*

*What was difficult about it? What did you do to overcome this difficulty?*

Throughout our teaching unit, we will use assessment regularly both to measure student progress, and to inform our teaching decisions. We will use the prompt above every day or every couple of days as needed. At least twice a week, we will give our students a small quiz consisting of two or three prompts, which will be graded according to a rubric. Each lesson will be concluded with an assignment to be worked on in groups in class or finished independently at home. These assignments will resemble the quizzes in the types of prompts given. We will see examples of these in the next section on learning activities.

At the end of the unit, a summative assessment will be given in which the students are required to work independently over two or three class sessions. The prompts given in this assessment will resemble those given in the quizzes and in the assignments. Below is a sample of questions that might appear on a summative assessment, complete with a rubric for each question and a discussion of how the prompts are designed to measure the students’ achievement of the goals given in Stage 1 of our design.

*Below (Figure 12) is a figure of a fuel gauge in a car. The scale from empty (E) to full (F) is divided into segments of equal size, and the arrow’s location on the scale shows how full the tank of gasoline is. Based on where the arrow is pointing, how much of a full tank do we have? How do you know?*
Explanation (1 point): 0 points if explanation is missing or does not make sense. 1 point if explanation refers to where the arrow lies along the length of the scale.

Numerical Answer (1 point): 0 points if answer is missing or if verbal or written expression is not equivalent to \( \frac{1}{4} \). 1 point if answer is equivalent to \( \frac{1}{4} \), including equivalent fractions, decimal expressions, or written out in words.

In addition to the emphasis on explanation, which is demonstration of the first facet of understanding, this problem also addresses working between representation modes. Specifically, the student is asked to translate a pictorial representation of one-fourth into a symbolic or verbal representation.

*You are making a batch of cookies. The recipe, which calls for 3 cups of flour, yields 60 cookies. You only want to make 30. How much flour do you use?*
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**Work Shown (1 point):** 0 points if no work is shown or is illegible. 1 point if work shown leads to answer given, even if the answer is incorrect.

**Numerical Answer (1 point):** 0 points if answer is missing or is incorrect. Half a point if answer is correct but no work is shown to support it. 1 point if answer is equivalent to \( \frac{1}{2} \) cups and the work shown supports this answer.

This problem requires students to recognize that less flour is required to make 30 cookies than what is required to make 60. The students must also choose which operation to use and on which quantities.

*Bar A and Bar B (Figure 13) are of equal length and width. Bar A is cut into four pieces of equal size while bar B is cut into six pieces of equal size as shown below.*

![Figure 13](image)

(a) What fraction of Bar A did you use to make your new bar?

(b) What fraction of Bar B did you use to make your new bar?
(c) Is your new bar as long as, shorter than, or longer than either Bar A or B used to be? How do you know?

(d) Add the fractions you wrote in parts (a) and (b). Write out the addition problem and show your work. What does your answer tell you about the length of your new bar as compared to the original length of either Bar A or Bar B? Explain.

Part (a): 1 point if the answer is \( \frac{3}{4} \) or some other equivalent fraction or phrase. 0 points otherwise.

Part (b): 1 point if the answer is \( \frac{1}{6} \) or some other equivalent fraction or phrase. 0 points otherwise.

Part (c): 2 points if the student indicates somehow that the new bar is shorter than either of the original bars and gives some explanation along the lines that one piece of Bar B is shorter than one piece of Bar A, so the new bar must be shorter. 1 point if the student simply says the new bar is shorter and provides no explanation or if the student says the new bar is longer and does provide an explanation. 0 points if this part is not done or if the answer is unintelligible.

Part (d): 3 points if the student writes an addition problem using the fractions obtained in parts (a) and (b), arrives at the correct sum, and gives an explanation expressing the idea that since \( \frac{11}{12} < 1 \), the new bar must be shorter. 2 points if the sum is calculated incorrectly but the conclusion and explanation are consistent with the sum found or the sum is calculated correctly and there is no conclusion and explanation given. 1 point if the sum is not correct and the conclusion and explanation are inconsistent with the sum. 0
points if the sum is not correct and no conclusion or explanation is provided or this part of the problem is not done.

Consider the subtraction problem \( \frac{5}{12} - \frac{1}{8} \). Pretend that there is a new student in our class, who is also in the fifth grade. In moving here from his home town across the country, he has missed most of the lessons on fractions, and does not know how to add or subtract them. Your job is to explain to him how to do this problem.

(a) Explain how to find a common denominator.

(b) Explain how to rewrite the fractions \( \frac{5}{12} \) and \( \frac{1}{8} \).

(c) Carry out the subtraction problem. Show your work.

**Part (a):** 2 points if the student explains that the product \( 12 \times 8 \) gives us a common denominator, or if the student tells us to make a list of multiples of 8 and another for 12 until we find a common multiple. 1 point if the student provides a correct common denominator but provides no explanation. 0 points otherwise.

**Part (b):** 2 points if the student explains that we multiply \( \frac{5}{12} \) by \( \frac{2}{2} \) (or multiply top and bottom by 2) and \( \frac{1}{8} \) by \( \frac{3}{3} \) (or multiply top and bottom by 3), for instance. 1 point if the student simply provides the correct equivalent fractions with the common denominator. 0 points otherwise.

**Part (c):** 2 points if the student shows the correct conversion of fractions into equivalent fractions with common denominator and carries out the subtraction correctly. 1 point if student makes a minor error. 0 points otherwise.
Stage 3: Designing Effective Learning Activities.

Now that we have articulated our learning goals and considered our methods for assessment, it is time to write the lesson plans and design the activities for this unit. First, let us discuss the sequence of instruction in this unit. One of our core tasks is the generation of equivalent fractions. This is an essential skill since students will be learning about adding, subtracting, and comparing fractions with unlike denominators for the first time. Under the Common Core, students have already been working with equivalent fractions since third grade. This work with equivalent fractions has focused mainly on the use of visual models of fractions to teach children that more than one symbolic representative is available for any given fraction. In the fourth grade, students learn that one can generate equivalent fractions by multiplying both the numerator and denominator of a fraction by the same natural number. That is, \( \frac{a}{b} = \frac{a \times n}{b \times n} \) for any natural number \( n \).

Where students will now need to generate equivalent fractions as part of a larger algorithm, such as in adding and subtracting fractions, we especially want students to understand that we multiply a given fraction by 1 in order to generate an equivalent fraction. That is, \( \frac{a}{b} = \frac{a \times n}{b \times n} \) for any natural number \( n \). For this reason, we will work with multiplication of fractions in our unit before we address addition and subtraction of fractions. We want students to be familiar with the algorithm for multiplication of fractions because we will be asking students to use this algorithm repeatedly in the generation of equivalent fractions.

We spend the first three lessons on multiplying fractions. From there, we move on to the concept of fraction equivalence, where we work with visual models first to gain an intuitive notion of fraction equivalence, and then work with algorithm of multiplication...
by 1 to generate equivalent fractions. There are three lessons on number sense, in which students learn to compare fractions and practice generating equivalent fractions. Then we treat addition and subtraction of fractions, mixed numbers, and converting from mixed number to improper fractions and vice versa. Finally we treat division of a natural number by a unit fraction, and division of a unit fraction by a natural number.

**Lesson 1: An area model for multiplying fractions.**

Objective: Students will work with the area of a unit square and the areas of partitions of a unit square in order to learn fraction multiplication.

Under the Common Core, students have been working with the notion of areas of rectangular regions since the second grade. In the third grade, students are formally taught that the area of a rectangular region is obtained by multiplying the length and the width of the region. We pose to our class an area problem upon which our lesson will be based.

Problem 1: A farmer owns a square-shaped region of land that is one mile long on each side. When he retires, he wants to leave the farm and his land to his six adult children. What is the area of the land the farmer owns now, and what is the area of the land each child will inherit?

We allow students to work in pairs and discuss this question for a couple of minutes. We discuss the answers as a class. Then we pose a follow-up question to our students and allow them five minutes to work on it.

Problem 2: You are a land surveyor. The farmer’s children have hired you to help them divide up the farm land their father has given them. Each child wants a rectangular piece
of land. Draw for them a picture of how you will divide up this land. Show them how long
and wide each piece of land will be.

Although there will be some variation among the pictures that students draw, Figure 14
shows two common responses we anticipate:

![Image of two common responses](image)

Figure 14

At this point, we ask the students if we can use the lengths and widths of these rectangular
regions to calculate their area. As a class, we have a discussion in which we guide the students
toward the conclusions that $1 \times \frac{1}{6} = \frac{1}{6}$ and $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. Next, we pose another follow-up problem.

**Problem 3:** After the land is all divided up among the six siblings, one of them decides to sell his
piece of land to his sister. His land is right next to his sister’s piece of land, so after she buys her
brother’s land, her piece of land is still rectangular. What is the area of her land now? Show her
share on the picture you drew earlier. How long and wide is the rectangle?
In Figure 15, we picture two possible responses. Again we discuss the question of area as a class. In each of the two scenarios pictured, the areas are the same. We guide the students to the conclusions that \(1 \times \frac{2}{6} = \frac{2}{6}\) and that \(\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}\). During this discussion, one of the students may propose a rectangular region of land for the sister that has dimensions \(1 \times \frac{1}{3}\). This is a good opportunity to refresh students’ memories of equivalent fractions from previous school years.

**Assessment:** Letting our students continue working in pairs, we ask students to partition squares to find the products \(\frac{1}{2} \times \frac{1}{4}\) and \(\frac{2}{3} \times \frac{3}{4}\). We circulate among the students to check for understanding and to answer questions.

**Lesson 2: An algorithm for multiplying fractions.**

**Objective:** Students will deduce an algorithm for multiplying two fractions.
We will provide students with a worksheet that guides them through some more examples of multiplying fractions by partitioning a square. The students may work in pairs. As the students work on these exercises, we circulate throughout the classroom to check for understanding and stop as necessary to discuss questions that come up frequently among the students. After the students have worked the six examples, we have a classroom discussion in which we try to come up with a rule that allows us to multiply fractions without drawing squares every time.

As we did in class yesterday, use the squares below to help you find the indicated products of fractions:

\[
\begin{align*}
\frac{1}{2} \times \frac{1}{2} \\
\frac{1}{4} \times \frac{1}{2}
\end{align*}
\]
After working the examples above, can you describe a rule for multiplying fractions that will not require us to draw a square every time we need to multiply two fractions?
Assessment: Apply your rule to the following products.

\[
\begin{array}{cccc}
\frac{1}{3} \times \frac{1}{6} & \frac{2}{3} \times \frac{2}{7} & \frac{3}{5} \times \frac{1}{4} & \frac{5}{8} \times \frac{3}{8}
\end{array}
\]

Without carrying out the multiplication, predict if multiplication by the second number will result in a product that is greater than, less than, or equal to the first number.

\[
\begin{array}{cccc}
\frac{5}{12} \times \frac{5}{9} & \frac{3}{7} \times \frac{4}{5} & \frac{1}{3} \times \frac{4}{4} & \frac{4}{3} \times \frac{1}{2}
\end{array}
\]

Lesson 3: Fractions as quotients and fractions as multiplicative quantities.

Objective: Students will learn that a fraction \( \frac{m}{n} \) can be interpreted as \( m \div n \) and can also be interpreted as \( m \times \frac{1}{n} \).

We introduce this idea with a problem. We have students working in pairs and we give them at least five minutes to come up with solutions.

Problem: How can you divide three loaves of bread evenly among four families? Working with your partner, draw a picture of three loaves of bread and show how you will cut up the bread to distribute to the four families.

We circulate among the students as they work on this problem, and we choose a response, similar to Figure 17, to share with the rest of the class.
In this first sample response, each color represents the portion of bread that goes to a family. In the class discussion, we want to emphasize that each piece of bread, which is any of the four parts of each loaf of bread, is the same size. We discuss the following questions

How many pieces of bread does each family receive?

How much of a loaf of bread is each piece?

How much of a loaf of bread does each family receive?

Using the answers to these questions, in the order they were given, we write \(3 \times \frac{1}{4} = \frac{3}{4}\).

We follow up this last sequence of questions with another sequence.

How many loaves of bread do we have?

Among how many families are we dividing the bread?

How much of a loaf of bread does each family receive?
Using the answers to these questions, in the order they were given, we write $3 ÷ 4 = \frac{3}{4}$.

Assessment: You are asked to divide two candy bars evenly among five children. How much of a candy bar does each child receive?

Write the fraction $\frac{2}{5}$ as a multiplication problem and as a division problem.

Write the fraction $\frac{4}{4}$ as a division problem. What is the quotient?

Write the fraction $\frac{4}{1}$ as a division problem. What is the quotient?

Lesson 4: Improper fractions, and multiplication of fractions by whole numbers.

Objective: Students will work within a real-world context to gain an intuitive understanding of improper fractions. Students will practice multiplying fractions by whole numbers.

We begin this lesson with a variation on the problem we examined in lesson three.

Problem 1: Suppose we have three loaves of bread that are cut into fourths as in yesterday’s problem. If I take five of the pieces, how much of a loaf of bread do I have?

We show Figure 18 to students.
The red pieces represent the portion I have taken. As we did in Lesson 3, we ask a series of questions, which are posted for all of the students to see. We have the students work in pairs for at least two minutes to discuss these questions.

*How many pieces of bread did I take?*

*How much of a loaf of bread is one piece?*

*How much of a loaf of bread did I take?*

For the third question, we anticipate that some students will propose the mixed number, $1 \frac{1}{4}$ loaves of bread, as the amount of bread I took, and we will acknowledge this answer as correct. But we are taking the class in a different direction. Using the answers to the first two questions, for which we expect little or no variation of responses, we write down for the class $5 \times \frac{1}{4}$. We ask...
the student about the pattern of fractions and their corresponding multiplication problems that they worked for the previous lesson. We want students to see that $5 \times \frac{1}{4} = \frac{5}{4}$. We press on.

*Which is more: $\frac{5}{4}$ of a loaf of bread, or 1 loaf of bread.*

We want students to see that there are fractions that are greater than 1. We then present students with a second problem, which is a slight variation on the previous problem. Again, we allow them to work in pairs for at least two minutes.

*Problem 2: You have three loaves of bread, and each loaf is cut into fourths. You and two of your friends each take two pieces. Answer the following questions:*

*How many people took bread?*

*What fraction of a loaf of bread did each of you take?*

*What fraction of a loaf of bread was taken in total?*

Using the answers in the order they were given, we arrive at $3 \times \frac{2}{4} = \frac{6}{4}$. Students may also answer the third question with the mixed number $1 \frac{2}{4}$ or $1 \frac{1}{2}$, or may even propose an alternative multiplication problem, $3 \times \frac{1}{2} = \frac{3}{2}$. We can put these alternative answers to good use in our next lesson on equivalent fractions, but for now, we want students to see the pattern of multiplication problem and product: $3 \times \frac{2}{4} = \frac{6}{4}$ and $3 \times \frac{1}{2} = \frac{3}{2}$. We will eventually want to link this pattern of multiplication problem and product to the algorithm for multiplying fractions, pointing out that we can also write $\frac{3}{1} \times \frac{1}{2} = \frac{3}{2}$, and we can help the students to establish this link in a homework assignment or other assessment.

*Assessment: Once again, you three loaves of bread, each cut into fourths. You and two of your friends each take three pieces.*

*Write a fraction that represents the portion of a loaf of bread that each of you took.*
Write a multiplication problem that shows how much of a loaf of bread all of you took together.

Exercises: Can you carry out the following multiplications without drawing loaves of bread?

\[
3 \times \frac{1}{7} \quad 4 \times \frac{2}{5} \quad 6 \times \frac{2}{3}
\]

How would you write 5 as a fraction? Remember that a fraction represents a division problem.

If you write 5 as a fraction, is this fraction an improper fraction? Why or why not?

Lesson 5: Equivalent fractions, part 1.

Objective: Students will begin to generate equivalent fractions.

We begin with a problem, and we allow students to work in pairs for at least two minutes before discussing the problem as a class.

Problem 1: Two friends have been operating a lemonade stand in their neighborhood. At the end of one day, they count the money they have made and set aside enough to buy more supplies for making lemonade. There is $16 left over.

If the friends split the money evenly between them, how much money does each one receive?

What fraction of the leftover money does each one receive?

We expect that not all answers will be the same. We center our discussion for the next few minutes around whether or not the different answers represent the same quantity. After discussing the students’ opinions for a time, we pose our next problem.

Problem 2: The same two friends who worked the lemonade stand later decide to bake a pan of brownies, which they will also share equally among themselves. See Figure 19.
The red part represents what one friend gets, the white part represents what the other friend gets. What fraction of the pan of brownies does each friend get?

The friends look at the pan of brownies and quickly decide that each share of brownies is too big to eat by themselves. They decide to cut their shares into smaller pieces, so they can share with their families, but they still want all pieces to be the same size. Here is what they try:
Figure 20
We ask the kids what fractions they can use to show what each friend’s share is. We hope that our students understand that it does not matter into how many pieces of equal size we cut the pan of brownies. Each friend will receive the same portion of the pan of brownies. Thus we must have that \( \frac{1}{2} = \frac{2}{4} = \frac{4}{8} \).

**Assessment:** We hand students a piece of paper with the most recent figure of the brownies shown on it. We explain to the students that each piece is still too big. We ask them to draw in how they would cut the pieces to make them smaller but still all the same size. Then we ask *What fraction of the pan does each friend receive? Can we write this fraction in terms of the number of white pieces and the total number of pieces? Is this fraction the same as \( \frac{1}{2} \)? Why or why not?*

**Lesson 6: Equivalent fractions, part 2.**

**Objective:** Students learn how to generate equivalent fractions of a given fraction by multiplying the fraction by an appropriate form of 1, where 1 is in the form of a fraction, \( \frac{n}{n} \) for some natural number \( n \).
We show the students Figure 21, and then hand each pair of students the following worksheet:

<table>
<thead>
<tr>
<th>Names:</th>
<th>Section 1</th>
<th>Section 2</th>
<th>Section 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>How many dots are green?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How many dots are there total?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>What fraction of the dots is green?</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We give the students at least two minutes to fill out the worksheet as we circulate among them to make sure they understand the task. When the students are done, and we have agreement among answers, we begin a class-wide discussion on the following question: *What patterns do you notice in the numbers as we move from Section 1, to 2, to 3?* Hopefully, the students will notice that the answers to the first two questions for Section 2 are twice those for Section 1, and that the answers to the first two questions for Section 3 are three times those for Section 1.

Then we ask *Are all the fractions the same?* With this question, we anticipate some disagreement among students. Some might claim that the fractions are not the same, for instance, arguing that \( \frac{3}{9} \) is greater than \( \frac{1}{3} \) because there are more dots in Section 3 than there are in Section 1. If and when a student makes this argument, we will distribute to the students dots made of construction paper. Each pair of students will receive three green dots and six blue dots. We will then ask the students to stack the three green dots on top of each other, so that they look like one green dot. Then we will ask them to create stacks of blue dots that are the same size as the stacks of green dots. Finally, we will ask the questions from the worksheet: *How many stacks are
green? How many stacks are there total? What fraction of the stacks is green? Some kids might begin to suspect that $\frac{1}{3} = \frac{3}{9}$.

To drive the point home, we next ask the pairs of students to work the following multiplication problems: $\frac{1}{3} \times \frac{2}{2}$ and $\frac{1}{3} \times \frac{3}{3}$. We give the students a minute or two to work these problems. Again, we ask if $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$. Why or why not? We want to see if a student points out that $\frac{2}{2} = \frac{3}{3} = 1$. Students already know that a number, when multiplied by 1, does not change. We want to see if any students make this connection, and we will guide the conversation in that direction.

Assessment:

For each of the fractions $\frac{1}{4}$ and $\frac{2}{3}$, find two more fractions that represent the same quantity.

Four people are playing a card game, and in this particular card game, the deck of cards has to be distributed among the players so that each player has the same number of cards. There are 52 cards in a deck of cards. What fraction of the deck does each player receive? How many cards does each player receive?

Lesson 7: An application of equivalent fractions.

Objective: Students will learn to compare two fractions with unlike denominators.

We begin with a problem to motivate the discussion.

Problem 1: Which fraction is greater: $\frac{2}{3}$ or $\frac{3}{5}$?

Math Trailblazers, a product of Teaching Integrated Mathematics and Science (TIMS), has a textbook for students in the fifth grade (Teaching Integrated Mathematics and Science [TIMS], 2003). The text recommends a pictorial approach and symbolic approach to comparing fractions. In the first approach, we use dot paper to construct rectangles that can be divided into fifths and
thirds. Students are to draw two such rectangles on the dot paper, shading two-thirds of one and three-fifths of another and comparing the areas that are shaded. See Figure 22.

Notice that the dimensions of each rectangle correspond to the denominators of the fractions we are comparing. We ask the students which rectangle has more shaded area. We hope that students identify the rectangle for $\frac{2}{3}$, reasoning that ten unit squares are shaded, while only nine are shaded in the rectangle depicting $\frac{3}{5}$. Thus, students conclude that $\frac{2}{3}$ is greater than $\frac{3}{5}$.

The second approach suggested by Math Trailblazers is that of finding a common denominator for the two fractions being compared, and converting the two fractions to equivalent fractions with that common denominator. The rectangles on the dot paper suggest how to do this,
so we ask the students to count the unit squares in each rectangle, both shaded and unshaded. They will answer that there are fifteen. How many unit squares are shaded in the rectangle for three-fifths, and how many in the rectangle for two-thirds? Students will answer ten and nine, respectively. We ask students to write equivalent fractions for \( \frac{2}{3} \) and \( \frac{3}{5} \) in terms of how many unit squares are shaded, and how many unit squares total there are in each rectangle. We call on students to explain to the class why \( \frac{3}{5} = \frac{9}{15} \) and \( \frac{2}{3} = \frac{10}{15} \).

Next, we ask the class if there is a way to arrive at the equivalent fractions without using the rectangles. How do we get from \( \frac{3}{5} \) to \( \frac{9}{15} \)? From \( \frac{2}{3} \) to \( \frac{10}{15} \)? We guide the discussion toward the conclusion that we multiply each fraction by an appropriate form of 1. That is, we end up multiplying the numerator and denominator of each fraction by the same natural number.

**Assessment:** We are comparing the fractions \( \frac{5}{8} \) and \( \frac{2}{3} \) to find out which one is greater. How do we determine the dimensions of the rectangles we will use for the comparison?

How do we compare the fractions \( \frac{1}{2} \) and \( \frac{5}{8} \) without drawing rectangles?

**Exercises:**

Determine which fraction is greater: \( \frac{3}{7} \) or \( \frac{1}{2} \).

Write equivalent fractions for \( \frac{3}{7} \) and \( \frac{1}{2} \) that have the same denominator.

Is there another way to compare the fractions \( \frac{3}{7} \) and \( \frac{1}{2} \) besides drawing the rectangles?

**Lesson 8: Adding and subtracting fractions.**

**Objective:** Students will perform addition and subtraction on fractions.
**Problem 1:** Two dogs, Zorro and Bear, always get fed at the same time. Zorro gets $\frac{1}{2}$ cup of dry food with each meal, while Bear gets $\frac{1}{3}$ cup. How much more food does Zorro get than Bear?

*How much dry food do the dogs eat at each meal?*

We have students work in pairs using the dot paper again. We ask them to concentrate on the first question only at this time. We suggest to the students that they use what they already know about comparing fractions to determine how much more food Zorro gets. We give the students a few minutes to talk within their pairs about how they would answer this question, then we convene as a class. We display the following figure for the students.

![Figure 23](image)

We ask students what the blue-shaded and the green-shaded portions of the rectangles represent. We want students to understand that the blue represents $\frac{1}{3}$ cup = $\frac{2}{6}$ cup of food, while the green represents $\frac{1}{2}$ cup = $\frac{3}{6}$ cup of food. We also check to make sure that students understand that the whole rectangle represents 1 cup of food. We expect that, after some discussion, the students will conclude from the picture that the extra amount that Zorro gets is represented by one green unit square, or $\frac{1}{6}$ cup of food. We ask if students can write a number sentence that
describes the relationship between the three fractions, and we want to students to see both
sentences, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, and its common-denominator equivalent, $\frac{3}{6} - \frac{2}{6} = \frac{1}{6}$.

So how much food do both dogs get? If no student suggests doing so, we suggest to the
class that they could try representing each dog’s portion in one rectangle. Students should then
see that the dogs eat $\frac{5}{6}$ cup at each meal. Then we ask if students can write another number
sentence that describes the relationship between the three fraction quantities. We ask for a
volunteer. We anticipate that we will either get the sentence $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, or $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$.

**Problem 2:** My brother and I buy two candy bars that are the same kind. He eats half of his and I
eat three-quarters of mine. How much candy bar have we eaten between the two of us? How
much more candy bar have I eaten than Bob? Be sure to use number sentences to answer both
questions.

We give the students several minutes to work on both questions by themselves. We circulate
among the students to check for examples and to choose volunteers to present their solutions to
the class. We expect that many will draw pictures on their dot paper that look similar to Figure
24.
We must be careful here to watch for students who mistake the augmented rectangle at the bottom of Figure 24 for one whole candy bar. From such students, we would expect answers like \( \frac{10}{10} \) or 1. We want to make sure that students understand that the \( 2 \times 4 \) rectangles in the figure above represent one whole candy bar. The \( 2 \times 5 \) rectangle, then, represents more than one whole candy bar. We will call on volunteers to explain to the class why \( \frac{3}{4} + \frac{1}{2} = \frac{6}{8} + \frac{4}{8} = \frac{10}{8} = 1 \frac{2}{8} \). We want students to develop intuition that tells them that since \( \frac{3}{4} > \frac{1}{2} \), we must have \( \frac{3}{4} + \frac{1}{2} \) is greater than 1. We make sure to have conversations with the class along those lines. As for the second question, we will look for students who show something like Figure 25:

![Figure 25](image)

We will call on such students to explain the solution to the second question to the class.

**Assessment:** Which is greater: \( \frac{4}{5} \) or \( \frac{3}{4} \)? What is their difference?

**Lesson 9: The role of common denominators in adding and subtracting fractions.**

**Objective:** Students will recognize that when two fractions have the same denominator, the sum or the difference of these fractions will also have the same denominator. They will recognize that numerator of the sum is the sum of the numerators, and that the numerator of the difference is
the difference of the numerators. Students will begin finding common denominators for fractions with unlike denominators.

**Discussion Question 1:** Consider the following number sentences that have come up as a result of our work with fractions:

\[
\frac{3}{6} + \frac{2}{6} = \frac{5}{6}, \quad \frac{3}{6} - \frac{2}{6} = \frac{1}{6}, \quad \frac{6}{8} + \frac{4}{8} = \frac{10}{8}, \quad \frac{6}{8} - \frac{4}{8} = \frac{2}{8}
\]

What patterns do you notice?

We approach this question using a think-pair-share strategy, in which we first ask students to think about the question on their own for a time. Then we have the students discuss the question with each other in pairs, and finally, we discuss the question as a class.

**Discussion Question 2:** Consider these number sentences that have also come up in our previous work with fractions:

\[
\frac{1}{2} + \frac{1}{3} = \frac{5}{6} \quad \text{and} \quad \frac{3}{4} + \frac{1}{2} = \frac{10}{8}
\]

\[
\frac{3}{6} + \frac{2}{6} = \frac{5}{6} \quad \text{and} \quad \frac{6}{8} + \frac{4}{8} = \frac{10}{8}
\]

How are the number sentences in the bottom line related to the number sentences in the top line?

Do you notice a pattern in the number sentences in the bottom line?

In the discussion of these two questions, we want the students to recognize the fractions in the bottom line as equivalent to the fractions in the top line. That is, \(\frac{3}{6} = \frac{1}{2}, \frac{2}{6} = \frac{1}{3}\), etc. We also want students to recognize the pattern of adding the numerators in the bottom line, while leaving the denominator unchanged.

Now we begin to guide the students toward finding common denominators.

**Discussion Question 3:** In the sums \(\frac{1}{2} + \frac{1}{3}\) and \(\frac{3}{4} + \frac{1}{2}\), why do you think that the fractions in the first sum were rewritten with a denominator of 6 while the fractions in the second sum were rewritten with a denominator of 8?
We expect that many students will identify the denominators 6 and 8 as the products of the two unlike denominators in each respective sum.

**Discussion Question 4:** How do we know to rewrite \(\frac{1}{2} + \frac{1}{3} \text{ as } \frac{3}{6} + \frac{2}{6}\)? Compare the figure below to Figure 23.

![Figure 26](https://example.com/figure26.png)

We want students to recognize that by drawing the dividing line across the middle of the left rectangle in Figure 26, we go from having three rectangular sub-regions to having six, as in Figure 23, and that we go from having one blue-shaded sub-region to having two, as in Figure 23. Thus \(\frac{1}{3} = \frac{2}{6}\). We want students to see that as we double the number of sub-regions, we also double the number of blue-shaded sub-regions. Symbolically, this doubling can be represented by multiplying each of the numerator and denominator of the fraction \(\frac{1}{3}\) by 2. Similarly, we can multiply each of the numerator and denominator of \(\frac{1}{2}\) by 3 to obtain \(\frac{1}{2} = \frac{3}{6}\).

**Assessment:** Fill in the missing fraction in each of the following number sentences to make the number sentence correct:

\[
\frac{1}{2} \times - = \frac{3}{6}, \quad \frac{1}{3} \times - = \frac{2}{6}.
\]
Explain how to do the following problem: $\frac{5}{8} - \frac{1}{2}$

**Lesson 10: Mixed numbers.**

**Objective:** Students will perform addition, subtraction, and multiplication of mixed numbers.

For addition and subtraction of mixed numbers, we will have students work with Pattern Blocks. Pattern blocks are manipulatives designed to help students model fractions with denominators of 2, 3, 4, 6, and 12 (TIMS, 2003). Figure 27 shows the Pattern Blocks.

![Pattern Blocks](image)

Figure 27

Pattern Blocks are used throughout the treatment of fractions in *Math Trailblazers*. Most notably, Pattern Blocks can be used to show fraction equivalency. For example, three of the green triangles can be arranged into the same shape as the red trapezoid. These arrangements can be stacked on top of one another to show equivalency. In particular, $3 \times \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$. But as we stated, we will be using them to help students see what happens when we add and subtract mixed numbers. We begin with an example.

**Problem 1:** Use Pattern Blocks to carry out the subtraction $1 - \frac{1}{3}$.

We offer students the hint that there is more than one way to represent 1 with the Pattern Blocks. The yellow hexagon, of course, represents 1. However, since we are subtracting from 1, we want
a representation of 1 from which we can take parts. We have our students work in pairs on this problem for a few minutes, and then we discuss possible solutions as a class. One possible student response is to form 1 with three blue rhombi. Taking away one rhombus from the formation, we would have two rhombi left, or $\frac{2}{3}$. Another possible response would be to take the red trapezoid as 1. We can assemble three green triangles into the same shape and size as the red trapezoid. Each green triangle would then represent one-third, and removing one of these green triangles from whole would then result in the expected $\frac{2}{3}$. Still another possible response would involve taking the brown trapezoid as 1 and the purple triangle as $\frac{1}{3}$.

**Problem 2:** Use Pattern Blocks to carry out the addition $1\frac{1}{2} + \frac{3}{4}$.

After having students work in pairs on this problem, we conduct a classroom discussion in which groups present their solutions. We work through a number of examples using the Pattern Blocks before asking students to talk about a method for adding and subtracting mixed numbers without the Pattern Blocks. We expect that students will suggest adding the whole numbers and the fractions separately. For example, in the problem $2\frac{1}{6} + 1\frac{3}{4}$, students will want to carry out $2 + 1$ and $\frac{1}{6} + \frac{3}{4}$ separately, to obtain $3\frac{11}{12}$. Of course, we encourage students to work with the Pattern Blocks to help them work out and explain what to do when the sum of the fractions in an addition problem is greater than 1, or when subtracting the fractions results in a difference that is less than 0.

**Problem 3:** Use Pattern Blocks to carry out the subtraction $3\frac{1}{4} - 2\frac{1}{2}$.

Now we move on to multiplication of mixed numbers.
Problem 4: In your backyard, you have set aside a rectangular region in which to create a flower bed. This region measures \(6\frac{1}{2}\) feet long by \(4\frac{3}{4}\) feet wide. What is the area of your flower bed?

We give students time to think about this question, expecting that many will suggest multiplying the whole numbers and fraction parts separately to obtain \(24\frac{3}{8}\). To show students why this method of multiplying mixed numbers does not work, we show students the following figure.

Carrying out the multiplication \(6\frac{1}{2} \times 4\frac{3}{4}\) involves calculating the areas of the four disjoint rectangles in Figure 28. We have \((6 \times 4) + \left(\frac{1}{2} \times 4\right) + \left(6 \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{3}{4}\right)\). Students are then left to find the four products and add them up. We have \(24 + 2 + \frac{18}{4} + \frac{3}{8} = 30\frac{7}{8}\). We have students practice on numerous examples of multiplication of mixed numbers, emphasizing the area model of multiplication.
Assessment: I write $\frac{2}{5} \times 1 \frac{5}{6} = \frac{10}{30}$. Explain why I am wrong, and show how to carry out the calculation.

Conclusion

Perhaps the most obvious weakness of this teaching unit is that it has not been tried on a class of fifth-graders. Are there better ways to conduct the learning activities we designed? Are there big ideas that we did not identify in the earlier stages of the design of this unit? Could we have encouraged deeper exploration of the connections between the big ideas? We believe that in order to answer these questions, we would have needed to teach this unit to several classes of fifth-graders. The reliability of our criteria for measuring our instructional success would necessarily depend on having taught this unit a number of times. Wiggins and McTighe point out that good rubrics take time to design. “Criteria and rubrics evolve with use. As you try them out, invariably you will find some parts of the rubric that work fine and some that don’t. Add and modify descriptions so that they communicate more precisely, and choose better anchors that illustrate what you mean” (Wiggins & McTighe, 2005, p. 182).

An interesting question to consider is whether or not fractions should be treated in a single teaching unit, or if subtopics within the domain of fractions can be treated within different units throughout the fifth grade mathematics curriculum. Math Trailblazers distributes content on fractions throughout the textbook (TIMS, 2003). This text introduces fractions in Unit 3, where the topics of part-to-whole relationships, equivalence of fractions, comparison of fractions, and rates are discussed. Unit 5 treats addition and subtraction of fractions by making use of visual and manipulative models before going on to devise symbolic algorithms. Units 11-13 develop fraction ideas further by showing how to divide numerator and denominator by common factors to reduce the terms of a fraction. Multiplication of fractions is introduced in
Unit 12 using visual models before progressing to developing a symbolic algorithm. Unit 13 develops the ideas of ratio and proportion, which are crucial to more advanced studies in algebra and geometry. Links between fractions and decimals are explored in Unit 7, and links between fraction and division are explored in Unit 9. Spreading out the treatment of fractions across teaching units in this way serves to emphasize the many and deep connections that fractions have to mathematical and real-world ideas.

Another interesting approach to teaching fractions is to develop fraction and decimal concepts concurrently, such as what is done in the Connected Mathematics 2 curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2009). Such an approach emphasizes the meaning of fraction and decimal representations of rational numbers, and the many connections that exist between the two. Algorithmically, arithmetic operations on fractions and decimals do not resemble each other closely. A curriculum that requires students to work within and between both representations of rational numbers is more likely to foster long-term recall of arithmetic algorithms than an approach that treats fraction representations and decimal representations as entirely separate entities.

Even though the effectiveness of our teaching unit has not been measured on an actual class of fifth-grade students, the considerations that went into the design of this unit should serve to underscore the importance of encouraging students to discover, for themselves, the many connections that exist between mathematical ideas. Otherwise, mathematics in general, and not just the topic of fractions, becomes a collection of unrelated and meaningless procedures. The ideas discussed in this paper are not new. Even so, traditional approaches to teaching mathematics, that emphasize procedure and memorization, prevail in many schools and classrooms in this country. However, as more and more people become concerned about how the
quality of American education compares with the quality of education in Asian countries, we feel that the approaches to teaching mathematics such as what we describe in this paper, and what are advocated and demonstrated in the TIMS and Connected Mathematics curricula, will attract greater attention from parents, teachers, and education policy-makers alike.
References


